Cluster theory of topological Fukaya categories

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Abstract

We establish a novel relation between the cluster categories associated with marked surfaces and the topological Fukaya categories of the surfaces. The Higgs category is a 2-Calabi–Yau extriangulated/exact ∞-category, generalizing the triangulated cluster category of the surface. We prove that the Higgs category is equivalent to the 1-periodic topological Fukaya category of the marked surface, as well as to the consingularity category of the relative Ginzburg algebra of the triangulated surface. The proof relies on the theory of perverse sheaves of categories on marked surfaces.

We furthermore give a general construction of 2-Calabi–Yau Frobenius extriangulated structures on stable ∞-categories equipped with a relative right 2-Calabi–Yau structure that may be of independent interest. When applied to the Higgs category of a surface, this yields a new construction of its extriangulated structure.

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Contents

1 Introduction

Cluster algebras are a class of commutative algebras equipped with generators exhibiting a remarkable combinatorial structure introduced by Fomin-Zelevinski [FZ02]. These generators are referred to as clusters and can be subjected to an operation called mutation, which produces other clusters. It was soon observed that cluster algebras admit a rich theory of categorification and this has lead to many fruitful interactions with the representation theory of algebras. A particular class of (additive) categorifications of cluster algebras are formed by so called cluster categories, which are triangulated 2-Calabi-Yau categories. Cluster categories of acyclic quivers were introduced in [BMR+06] as certain orbit categories. A generalization to arbitrary quivers with potential was achieved by Amiot [Ami09]. Starting with a quiver with potential (Q,W), we can consider the perfect derived category of the 3-Calabi-Yau Ginzburg dg-algebra $\mathscr{G}(Q,W)$. Amiot's generalized cluster category assigned to $\mathscr{G}(Q,W)$ is then defined as the cosingularity category

$$\mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,W))/\mathcal{D}^{\mathrm{fin}}(\mathscr{G}(Q,W))\,,$$

meaning the Verdier quotient of the perfect derived category by the finite derived category. The generalized cluster category is triangulated 2-Calabi–Yau and admits cluster tilting objects [Ami09].

Recently, generalizations of Ginzburg algebras have emerged [Wu23, Chr22a], called relative Ginzburg algebras, associated with ice quivers with potentials. An interesting class of relative Ginzburg algebras, studied by the author in [Chr22a, Chr21], arises from marked surfaces equipped with an ideal triangulation. In this paper, we suppose that the marked surfaces has no marked points in the interior, i.e. no punctures. We denote by $\mathscr{G}_{\mathcal{T}}$ the relative Ginzburg algebra associated with a marked surface $\mathbf{S}$ equipped with an ideal triangulation $\mathcal{T}$. The main result of this article can be stated as follows:

Theorem 1 (Theorem 8.4, Theorem 4.8)**.** *Let $\mathbf{S}$ be a marked surface with an ideal triangulation $\mathcal{T}$. The following three ∞-categories are equivalent:*

(1) The Higgs category $\mathcal{H}$ of [Wu23] associated with the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$.

(2) The cosingularity category $\mathcal{D}^{\mathrm{perf}}(\mathscr{G}_{\mathcal{T}})/\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})$ of the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$.

(3) The topological Fukaya category of $\mathbf{S}$ valued in the 1-periodic perfect derived category $\mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}])$, denoted $\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$.

The equivalence between (1) and (3) furthermore extends to an equivalence of associated exact ∞-structures.

We give a few comments on Theorem 1:

- The Higgs category, introduced by Yilin Wu in [Wu23], is a relative version of the cluster category of [Ami09] and can be associated with the relative Ginzburg algebra of an ice quiver with potential. Instead of being triangulated, it is a Frobenius extriangulated category, in the sense of [NP19]. It is furthermore extriangulated 2-Calabi–Yau and admits cluster tilting objects. It can be used to categorify cluster algebras *with coefficients*, where the coefficients correspond to the frozen vertices of the ice quiver. The Higgs category admits a canonical enhancement to an exact ∞-category which Theorem 1 refers to.
- The topological Fukaya category of an oriented marked surface with coefficients in a 2-periodic dg category was constructed in [DK18]. Theorem 1 refers to the stable ∞-categorical avatar of this topological Fukaya category. As coefficients, we consider the perfect derived category of 1-periodic chain complexes over a base field k. Theorem 1 is shown under the assumption that k is algebraically closed and $\mathrm{char}(k) \neq 2$.
- Higgs categories associated with ice quivers typically do not admit a triangulated structure. A surprising consequence of Theorem 1 is that the Higgs categories of surfaces admit such triangulated structures. We emphasize that this triangulated structure is distinct from its extriangulated structure. We also give an alternative description of the extriangulated structure in terms of the topological Fukaya category, see below.

As shown in [Wu23], the stable category of the Frobenius extriangulated Higgs category of an ice quiver recovers the generalized cluster category of the non-frozen part of the quiver. As a consequence of Theorem 1, we thus relate the 1-periodic topological Fukaya category with the usual triangulated cluster category of the surface $C^{\mathrm{tri}}_{\mathbf{S}}$ associated with the quiver with potential of the surface of [LF09]:

Corollary 1. *The stable category $\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ of the Frobenius exact ∞-category $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ is an enhancement of the triangulated cluster category $C^{\mathrm{tri}}_{\mathbf{S}}$ associated with* $\mathbf{S}$. *Stated differently, there exists an equivalence of triangulated categories* $\mathrm{ho}\,\bar{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}} \simeq C^{\mathrm{tri}}_{\mathbf{S}}$.

Corollary 1 establishes a precise and novel link between cluster categories of surfaces and topological Fukaya categories, and thus by extension with the derived categories of gentle algebras, see also Proposition 4.16 and [Chr25b, Prop. 4.11].

An application of Corollary 1 is the following: the 1-periodic topological Fukaya category $\mathcal{C}_{\mathbf{S}}$ admits an action of the mapping class group of the surface by [DK18], see also Theorem 4.15. This mapping class group action localizes to an action on the triangulated cluster category $C^{\mathrm{tri}}_{\mathbf{S}}$, see Proposition 8.9. Such an action has not been previously constructed.

Cosingularity categories and quotient perverse schobers

The proof of the equivalence of (2) and (3) in Theorem 1 relies on the theory of perverse schobers parametrized by ribbon graphs of [Chr22a]. The existence of a notion of perverse schobers was proposed by Kapranov–Schechtman [KS14]. A perverse schober refers loosely speaking to a perverse sheaf of enhanced triangulated categories. The parametrized perverse schobers considered in this paper amount to certain constructible sheaves of stable ∞-categories, defined on a spanning graph of the marked surface.

As shown in [Chr22a], the derived ∞-category $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ of the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ arises as the global sections of such a parametrized perverse schober, denoted $\mathcal{F}_{\mathcal{T}}$. We show that the passage to the (Ind-completed) cosingularity category $\mathcal{D}(\mathscr{G}_{\mathcal{T}})/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})$ lifts to the level of perverse schobers:

Theorem 2 (Theorem 4.8, Proposition 4.13)**.** *The stable ∞-categories $\mathcal{C}_{\mathbf{S}}$ and $\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})$ arise as the global sections of two perverse schobers $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ and $\mathcal{F}^{\mathrm{mnd}}_{\mathcal{T}}$. These fit into a fiber and cofiber sequence of perverse schobers:*

$$\begin{tikzcd} \mathcal{F}^{\mathrm{mnd}}_{\mathcal{T}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{F}_{\mathcal{T}} \arrow[d] \\ 0 \arrow[r] & \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}} \end{tikzcd} \tag{1}$$

The notion of a fiber and cofiber sequence of parametrized perverse schobers is introduced in Section 3.3.

Parametrized perverse schobers are locally described by spherical adjunctions. For $\mathcal{F}_{\mathcal{T}}$, there is one such spherical adjunction for each triangle of the ideal triangulation. These adjunctions are each given by

$$\phi^*\colon \mathcal{D}(k) \longleftrightarrow \mathcal{D}(k[t_1]) : \phi_*\,,$$

where $k[t_1]$ denotes the graded polynomial algebra with generator in degree 1 and ϕ is the morphism of dg-algebras $k[t_1] \xrightarrow{t_1\mapsto 0} k$.

The spherical adjunctions underlying $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$ are all given by

$$\mathcal{D}(k) \longleftrightarrow \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1]) \tag{2}$$

arising from restricting $\phi^* \dashv \phi_*$. The perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ is consequently described by the 'quotient' spherical adjunction

$$0 \longleftrightarrow \mathcal{D}(k[t_1])/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_1]) \simeq \mathcal{D}(k[t_1^{\pm}])\,,$$

which is the trivial spherical adjunction with the 1-periodic derived ∞-category $\mathcal{D}(k[t_1^{\pm}])$. Indeed, topological Fukaya categories arise as the global sections of perverse schobers with such trivial underlying spherical adjunctions.

The passage to the cosingularity category has a curious interpretation in terms of the theory of perverse schobers: we will see that the spherical adjunction (2) is the monadic adjunction associated to the adjunction monad $\phi_*\phi^*$ of the adjunction $\phi^* \dashv \phi^*$. We expect the fiber and cofiber sequence of Theorem 2 to be a special case of a general principle, by which, under mild assumptions, a perverse schober can be decomposed into perverse schobers with monadic or trivial spherical adjunctions.

Work of Ivan Smith [Smi15] gives a symplectic interpretation of the finite derived category of the (non-relative) Ginzburg algebra associated with a triangulated marked surface: its homotopy category embeds fully faithfully into the derived Fukaya category of a Calabi–Yau threefold Y with a Lefschetz fibration π to $\mathbf{S}$. The wrapped Fukaya category of the regular fiber of π is equivalent to $\mathcal{D}^{\mathrm{perf}}(k[t_1])$. From this perspective, the 1-periodic derived category $\mathcal{D}(k[t_1^{\pm}])$ describes the Rabinowitz wrapped Fukaya category of this fiber, which is by [GGV22] equivalent to Efimov's (algebraizable) formal punctured neighborhood of infinity [Efi17] of the wrapped Fukaya category of the fiber, see also Proposition 2.12.

Categorical cluster theory

An additive categorification of a cluster algebra consists of a triangulated or extriangulated 2-Calabi–Yau category, equipped with cluster tilting objects. Cluster tilting objects admit a well behaved theory of mutations, see [IY08], that categorifies the mutations of clusters in the cluster algebra. For an additive categorification of a cluster algebra one asks that the cluster tilting objects in $\mathcal{C}$ are in bijection with the clusters of the cluster algebra. There should further be a decategorification map, called a cluster character, which associates elements of the cluster algebra with objects of the category, and satisfying different properties, see Definition 7.20 or [Pal08, KW23].

To explain how the 1-periodic topological Fukaya category fits into this picture, we begin by explaining the origin of its 2-Calabi–Yau extriangulated structure. The topological Fukaya category $\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ comes equipped with a functor

$$G = \prod_{e\in \mathcal{T}_1^{\partial}} \mathrm{ev}_e\colon \mathcal{C}_{\mathbf{S}} \longrightarrow \prod_{e\in\mathcal{T}_1^{\partial}} \mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}])\,, \tag{3}$$

which takes a global section and evaluates it at all the external edges of the dual trivalent graph of $\mathcal{T}$ (the set of which is denoted by $\mathcal{T}_1^{\partial}$). This dual graph is the graph parametrizing $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$. A fiber and cofiber sequence in $\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ is declared to be exact if its image under G splits, and this gives rise to the structure of a Frobenius exact ∞-category, see Proposition 5.9. The homotopy category of $\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ thus inherits an extriangulated structure.

The category $\mathcal{C}_{\mathbf{S}}$ is smooth and proper as a 2-periodic, meaning $k[t_2^{\pm}]$-linear, stable ∞-category. Using results on the gluing of relative Calabi–Yau structures and perverse schobers from [Chr26], we find that the $k[t_2^{\pm}]$-linear functor G carries a relative right 2-Calabi–Yau structure. This is a 2-periodic and proper version of the result of [BD19] on relative left Calabi–Yau structures on topological Fukaya categories. We show in Propositions 5.19 and 5.21 in a more general setup that this implies that the extriangulated structure on $\mathrm{ho}\,\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ is 2-Calabi–Yau.

Next, we turn to the description of the cluster tilting objects in $\mathrm{ho}\,\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$. We classify the cluster tilting objects in $\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ as follows:

Theorem 3 (Combine [FT18, Theorem 8.6] and Corollary 7.15)**.** *There are canonical bijections between the sets of the following objects.*

- *Clusters of the cluster algebra with coefficients in the boundary arcs associated with* $\mathbf{S}$*.*
- *Ideal triangulations of* $\mathbf{S}$*.*
- *Cluster tilting objects in the* 2*-Calabi–Yau Frobenius extriangulated category* $\mathrm{ho}\,\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$*.*

To prove Theorem 3, we classify the indecomposable objects of $\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ in terms of curves in the surface $\mathbf{S}$ and describe their exact extensions in terms of crossing. For this, we use the results of [Chr21], which associates global sections of $\mathcal{F}_{\mathcal{T}}$ with so-called matching curves in the surface. We prove that every object in $\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ arises this way:

Theorem 4 (The geometrization Theorem 6.13)**.** *Every object $X \in \mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$ of the* 1*-periodic topological Fukaya category splits uniquely into a finite direct sum of indecomposable objects associated with matching curves.*

A similar classification result for the objects in the generalized cluster categories of non-relative Ginzburg algebras arising from triangulated surfaces without punctures appears in [BZ11]. The case with punctures was treated in [AP21, QZ17]. The novelty in the proof of Theorem 4 is that it bases on ∞-categorical gluing techniques and in particular the description of objects in limits of diagrams of ∞-categories in terms of sections of the Grothendieck construction.

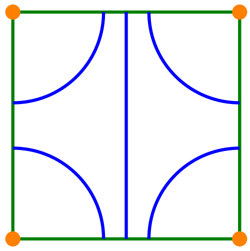
As shown in [Mul16], the commutative Kauffman skein algebra $\mathrm{Sk}_{\mathbf{S}}^{1,\mathrm{loc}}$ of the surface $\mathbf{S}$ (with parameter $q = 1$ and invertable boundary arcs) embeds into the upper cluster algebra of the surface $\mathbf{S}$ with coefficients in the boundary arcs, see also Theorem 7.13 for a precise statement. The skein algebra is generated by links in $\mathbf{S}$, meaning superpositions of curves, modulo relations such as the Kauffman skein relation, see Definition 7.10. Combining this fact with Theorem 4, we can describe an explicit cluster character to the upper cluster algebra:

Theorem 5 (Theorem 7.21)**.** *There is a cluster character $\chi\colon \mathrm{obj}(\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}) \to \mathrm{Sk}_{\mathbf{S}}^{1}$ with values in the commutative skein algebra of links in* $\mathbf{S}$*. The character maps an object corresponding to a matching curve in* $\mathbf{S}$ *to the matching curve considered as a link with a single component. Furthermore, composing χ with the map from $\mathrm{Sk}_{\mathbf{S}}^{1}$ to the upper cluster algebra yields a cluster character to the upper cluster algebra.*

We use Theorem 3 to prove the equivalence of (1) and (2)/(3) in Theorem 1 as follows: there is a canonical exact functor $\mathcal{H} \to \mathcal{D}^{\mathrm{perf}}(\mathscr{G}_{\mathcal{T}})/\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})$ to the consingularity category, which we show to map an initial cluster tilting object to a cluster tilting object. Using that cluster tilting objects are particularly strong generators, this allows under mild conditions to deduce that the functor is an equivalence of categories, see Lemma 8.6.

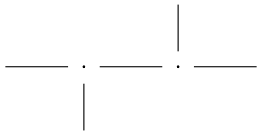
Finally, we illustrate our results in the simple case of $\mathbf{S}$ being the 4-gon.

Example 1. Let $\mathbf{S}$ be the 4-gon, depicted as follows, with an ideal triangulation (in blue).



(4)

The corresponding dual ribbon graph has two vertices and can be depicted as follows:



At each vertex, the spherical adjunction underlying $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ is $0 \leftrightarrow \mathcal{D}(k[t_1^{\pm}])$, where $\mathcal{D}(k[t_1^{\pm}]) \simeq \mathcal{D}(k)/[1]$ is the unbounded 1-periodic derived category and equivalent to the ∞-categorical orbit category, see [Chr25b].

This underlying spherical adjunction determines the value of $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ at each of the two vertices: the stalks are equivalent to $\mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}])) \simeq \mathcal{D}(kA_2)/[1]$, where kA_2 is the path algebra of the A_2-quiver, see the local model for perverse schobers recalled in (7).

Passing to global sections, we glue two A_2 quivers to an A_3-quiver, and obtain that the (Ind-complete) 1-periodic topological Fukaya category is given by $\mathcal{D}(kA_3)/[1]$. Note that the path algebra kA_3 is the gentle algebra associated with a 4-gon on the sense of [OPS18]. The Higgs ∞-category is thus by Theorem 1 equivalent to $\mathcal{D}^{\mathrm{perf}}(kA_3)/[1]$, and additionally equipped with a non-trivial exact ∞-structure.

Each arc in the 4-gon defines an object of this Higgs category and a cluster tilting object is given by sum of the objects arising from an ideal triangulation. The (discrete) endomorphism algebra of the cluster tilting object is called the Jacobian algebra. The Jacobian algebra of the ideal triangulation (4) is gentle and given by the quotient of the path algebra of the quiver

$$\begin{tikzcd}
1 \arrow[dd, "a"'] & & 2' \arrow[dl, "b'"'] \\
& 3 \arrow[ul, "c"'] \arrow[dr, "c'"] & \\
2 \arrow[ur, "b"] & & 1' \arrow[uu, "a'"']
\end{tikzcd}$$

by the ideal $(ba, cb, ac, b'a', c'b', a'c')$. The Jacobian algebras of arbitrary ideal triangulations are also gentle, see Proposition 7.17.

Notations and conventions

In this paper, we freely employ the language of ∞-categories, as developed in [Lur09, Lur17]. We generally follow the notation and conventions of [Lur09, Lur17]. We use the homological grading. Given an ∞-category $\mathcal{C}$ and two objects $X, Y \in \mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the mapping space. We further use the following conventions.

- We denote the homotopy 1-category of an ∞-category $\mathcal{C}$ by $\mathrm{ho}\,\mathcal{C}$.
- We denote the right adjoint of a functor F by $\mathrm{radj}(F)$ and the left adjoint by $\mathrm{ladj}(F)$ (if existent).
- Given a functor $F\colon \mathcal{A} \to \mathcal{B}$, we denote by $F(\mathcal{A})$ the essential image of F, meaning the smallest full subcategory of $\mathcal{B}$ which is closed under equivalences in $\mathcal{B}$ and contains the objects in the image of F.

- Matrices can be used to describe morphisms between direct sums of objects in additive ∞-categories, such as stable ∞-categories. For this we use the convention of multiplication with a matrix from the left.

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2 Preliminaries

2.1 ∞-categories of ∞-categories

We use the following notation for different ∞-categories of ∞-categories. We denote by

- $\mathcal{S}$ the ∞-category of spaces.
- Cat_∞ the ∞-category of ∞-categories.
- St the ∞-category of stable ∞-categories and exact functors.
- $\mathcal{P}r^L$ the ∞-category of presentable ∞-categories and left adjoint functors.
- $\mathcal{P}r^R$ the ∞-category of presentable ∞-categories and right adjoint functors.
- by $\mathcal{P}r^L_{\mathrm{St}} \subset \mathcal{P}r^L$ and $\mathcal{P}r^R_{\mathrm{St}} \subset \mathcal{P}r^R$ the full subcategories consisting of stable ∞-categories.

We remark that passing to adjoint functors defines an equivalence of ∞-categories $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\mathrm{op}}$, see [Lur09, 5.5.3.4].

Let $\mathcal{C}$ be an ∞-category. An object x in an ∞-category $\mathcal{C}$ is called compact if the functor $\mathrm{Map}_{\mathcal{C}}(x,\text{-})\colon \mathcal{C} \to \mathcal{S}$ preserves filtered colimits. We denote by $\mathcal{C}^{\mathrm{c}} \subset \mathcal{C}$ the full subcategory of compact objects. If $\mathcal{C}$ is small, we denote by $\mathrm{Ind}(\mathcal{C})$ the Ind-completion of $\mathcal{C}$. If $\mathcal{C}$ is small, stable and idempotent complete, then $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C})^{\mathrm{c}}$.

The ∞-category $\mathcal{P}r^L$ admits a symmetric monoidal structure, such that a commutative algebra object in $\mathcal{P}r^L$ amounts to a symmetric monoidal presentable ∞-category $\mathcal{C}$, satisfying that its tensor product $\text{-}\otimes\text{-}\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ preserves colimits in both entries, see [Lur17, Section 4.7]. An example of such a commutative algebra object in $\mathcal{P}r^L$ is the derived ∞-category $\mathcal{D}(R)$ of a commutative dg-algebra R.

Definition 2.1. Let R be a commutative dg-algebra. We call the ∞-category $\mathrm{Mod}_{\mathcal{D}(R)}(\mathcal{P}r^L)$ of modules, see [Lur17, Def. 4.5.1.1], the ∞-category of R-linear presentable ∞-categories and denote it by LinCat_R.

As noted in [Lur18, Section D.1.5], presentable R-linear ∞-categories are automatically stable.

Definition 2.2 ([Lur17, Def. 4.2.1.28])**.** Let R be a commutative dg-algebra. Let $\mathcal{C}$ be an R-linear ∞-category and let $X,Y \in \mathcal{C}$. A morphism object for X,Y is an R-module $\mathrm{Mor}_{\mathcal{C}}(X,Y) \in \mathcal{D}(R)$ equipped with a map $\alpha\colon \mathrm{Mor}_{\mathcal{C}}(X,Y)\otimes_R X \to Y$ in $\mathcal{C}$ such that for every object $C \in \mathcal{D}(R)$ composition with α induces an equivalence of spaces

$$\mathrm{Map}_{\mathcal{D}(R)}(C, \mathrm{Mor}_{\mathcal{C}}(X,Y)) \to \mathrm{Map}_{\mathcal{C}}(C\otimes_R X, Y)\,.$$

We thus have $\mathrm{H}_i \mathrm{Mor}_{\mathcal{C}}(X,Y) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(X[i],Y)$.

Remark 2.3. Morphism objects in any presentable R-linear ∞-category $\mathcal{C}$ exist by [Lur17, Prop. 4.2.1.33] and the formation of morphism objects forms a functor

$$\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \longrightarrow \mathcal{D}(R)\,,$$

see [Lur17, Rem. 4.2.1.31].

Given an R-linear stable ∞-category $\mathcal{C}$ and two objects $A,B\in\mathcal{C}$, the n-th Ext-group is defined as

$$\mathrm{Ext}^n_{\mathcal{C}}(A,B) \coloneqq \mathrm{H}_{-n}\mathrm{Mor}_{\mathcal{C}}(A,B) \simeq \mathrm{H}_0 \mathrm{Mor}_{\mathcal{C}}(A,B[n])\,.$$

2.2 Semiorthogonal decompositions and Verdier quotients

Given a stable ∞-category $\mathcal{C}$ and a full subcategory $\mathcal{A} \subset \mathcal{C}$, we call $\mathcal{A}$ a stable subcategory of $\mathcal{C}$ if $\mathcal{A}$ is stable, the inclusion is an exact functor and $\mathcal{A}$ is closed under equivalences in $\mathcal{C}$.

Given a fully faithful functor $i\colon \mathcal{A} \to \mathcal{C}$, we will later on sometimes abuse notation by identifying $\mathcal{A}$ with the stable subcategory of $\mathcal{C}$ given by the essential image of i.

Definition 2.4 ([Lur18, Def. 7.2.0.1])**.** Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ be stable subcategories. We say that the pair $(\mathcal{A}, \mathcal{B})$ forms a semiorthogonal decomposition of $\mathcal{C}$ if

- for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(b, a)$ is contractible and
- for all $x \in \mathcal{C}$, there exists a fiber and cofiber sequence $b \to x \to a$ in $\mathcal{C}$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

We will be interested in semiorthogonal decompositions of stable and presentable ∞-categories. A fiber and cofiber square in $\mathcal{P}r^L_{\mathrm{St}}$, which can be depicted as follows,

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{B} \end{tikzcd}$$

is also called an *exact sequence* of stable, presentable ∞-categories.

Remark 2.5. Given a fiber and cofiber sequencce in $\mathcal{P}r^L_{\mathrm{St}}$ as above, the triangulated homotopy category of $\mathcal{B}$ is equivalent to the triangulated Verdier quotient of $\mathcal{C}$ by $\mathcal{A}$, see [BGT13, Prop. 5.9]. We thus call $\mathcal{B}$ the Verdier quotient of $\mathcal{C}$ by $\mathcal{A}$.

The following Lemma shows that the datum of a semiorthogonal decomposition of a stable, presentable ∞-category is equivalent to the datum of a fiber and cofiber sequence in $\mathcal{P}r^L_{\mathrm{St}}$.

Lemma 2.6. *Consider a diagram in* $\mathcal{P}r^L_{\mathrm{St}}$

$$\mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi} \mathcal{B}\,. \tag{5}$$

Suppose that i and $\mathrm{radj}(\pi)$ are fully faithful. Then the diagram can be extended to a fiber and cofiber square if and only if $(\mathrm{radj}(\pi)(\mathcal{B}), i(\mathcal{A}))$ forms a semiorthogonal decomposition of $\mathcal{C}$.

Proof. Suppose that (5) is part of a fiber and cofiber square. We find that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(\iota(a), \mathrm{radj}(\pi)(b)) \simeq \mathrm{Map}_{\mathcal{B}}(\pi\iota(a), b) \simeq \mathrm{Map}_{\mathcal{B}}(0, b)$ is contractible. Let $c \in \mathcal{C}$ and $\mathrm{cu}_c\colon c \to \mathrm{radj}(\pi)(\pi(c))$ be the counit. The map $\pi(\mathrm{cu}_c)$ is an equivalence by the 2/3-property and the fact that $\mathrm{radj}(\pi)$ is fully faithful. We hence have $\pi(\mathrm{fib}(\mathrm{cu}_c)) \simeq 0$. It follows that $\mathrm{fib}(\mathrm{cu}_c) \in \mathrm{Im}(i)$. There thus exists an object $a \in \mathcal{A}$ and an exact sequence $a \to c \to \mathrm{radj}(\pi) \circ \pi(c)$ in $\mathcal{C}$. We have shown that $(\mathrm{radj}(\pi)(\mathcal{B}), i(\mathcal{A}))$ forms a semiorthogonal decomposition of $\mathcal{C}$.

For the converse implication, assume that $(\mathrm{radj}(\pi)(\mathcal{B}), i(\mathcal{A}))$ forms a semiorthogonal decomposition of $\mathcal{C}$. The fiber of π is the full subcategory of $\mathcal{C}$ spanned by objects $c \in \mathcal{C}$, satisfying that $\mathrm{Map}_{\mathcal{C}}(c, \mathrm{radj}(\pi)(b))$ is contractible for all $b \in \mathcal{B}$. Proposition 2.2.4 from [DKSS21] shows this subcategory agrees with $i(\mathcal{A})$ since $(\mathrm{radj}(\pi)(\mathcal{B}), i(\mathcal{A}))$ forms a semiorthogonal decomposition. To show that $\mathcal{B}$ is the cofiber of i, we use the equivalence $\mathrm{radj}\colon \mathcal{P}r^L_{\mathrm{St}} \simeq (\mathcal{P}r^R_{\mathrm{St}})^{\mathrm{op}}$ and show there exists a fiber sequence in Cat_∞ as follows:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathrm{radj}(\pi)"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d, "\mathrm{radj}(i)"] \\ 0 \arrow[r] & \mathcal{A} \end{tikzcd}$$

This is in turn equivalent to the assertion that $\mathrm{radj}(\pi)$ defines an equivalence between $\mathcal{B}$ and the full subcategory of $\mathcal{C}$ spanned by objects $c \in \mathcal{C}$, satisfying that $\mathrm{Map}_{\mathcal{C}}(i(a), c)$ is contractible for all $a \in \mathcal{A}$. Proposition 2.2.4 from [DKSS21] again guarantees this. $\square$

2.3 Monadicity and derived categories of graded Laurent algebras

Let $\mathcal{C}$ be an ∞-category. A monad M on $\mathcal{C}$ is an associative algebra object in the monoidal ∞-category $\mathrm{End}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C},\mathcal{C})$ of endofunctors. In other words, a monad is a functor $M\colon \mathcal{C} \to \mathcal{C}$, together with a unit $u\colon \mathrm{id}_{\mathcal{C}} \to M$ and a multiplication map $M \circ M \to M$ equipped with data exhibiting coherent associativity and unitality. A main source of monads are adjunctions: given an adjunction $F\colon \mathcal{C} \leftrightarrow \mathcal{D} : G$ of ∞-categories, there is an associated monad $M = GF$, see [Lur17, Section 4.7.3], which we call the adjunction monad. Associated to a monad M on $\mathcal{C}$ is its ∞-category $\mathrm{LMod}_M(\mathcal{C})$ of left modules in $\mathcal{C}$ and the free-forget adjunction

$$\mathrm{Free}\colon \mathcal{C} \longleftrightarrow \mathrm{LMod}_M(\mathcal{C}) : \mathrm{Forget}\,,$$

whose adjunction monad is equivalent to M, see [Lur17]. The ∞-category $\mathrm{LMod}_M(\mathcal{C})$ is also called the Eilenberg-Moore ∞-category of M. A typical example is given by $\mathcal{C} = \mathcal{D}(k)$ the derived ∞-category of a field and $M = A \otimes_k$ - the monad arising from tensoring with a dg-algebra A. In this case $\mathrm{LMod}_M(\mathcal{D}(k)) \simeq \mathcal{D}(A)$ and associated adjunction is equivalent to the usual free-forget adjunction $\mathcal{D}(k) \leftrightarrow \mathcal{D}(A)$.

Given an adjunction $F\colon \mathcal{C} \leftrightarrow \mathcal{D} : G$ with monad $M = GF$, there is an associated functor $\mathcal{D} \to \mathrm{LMod}_M(\mathcal{C})$, making the following diagram commute:

$$\begin{tikzcd} \mathcal{D} \arrow[rr] \arrow[dr, "G"'] & & \mathrm{LMod}_M(\mathcal{C}) \arrow[dl, "\mathrm{Forget}"] \\ & \mathcal{C} & \end{tikzcd}$$

Definition 2.7. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories.

(1) The functor G is called monadic if it admits a left adjoint F with adjunction monad $M = GF$ and the associated functor $\mathcal{D} \to \mathrm{LMod}_M(\mathcal{C})$ is an equivalence of ∞-categories.

(2) The functor G is called comonadic if the opposite functor $G^{\mathrm{op}}\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is monadic.

If $\mathcal{C},\mathcal{D}$ are presentable ∞-categories and G preserves (sufficient) colimits, then $\mathcal{D} \to \mathrm{LMod}_M(\mathcal{C})$ admits a fully faithful left adjoint, see [Lur17, Lemma 4.7.3.13]. In this case, $\mathrm{LMod}_M(\mathcal{C})$ is presentable and if $\mathcal{C}$ and $\mathcal{D}$ are furthermore stable, then $\mathrm{LMod}_M(\mathcal{C})$ is also stable, see [Lur17, Prop. 4.2.3.4].

To determine whether a functor is monadic, one can use the Lurie–Barr–Beck monadicity theorem, see [Lur17, Thm. 4.7.3.5]. Assuming that all involved ∞-categories are presentable and that the functor preserves colimits, the theorem reduces to the statement that a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ is monadic if and only if it is conservative, i.e. reflects isomorphisms. Similarly, in this setting a left adjoint F which preserves sufficient limits is comonadic if and only if it is conservative.

We now turn to a specific example of a monadic adjunction. Let k be a field and $n \geq 2$. We are interested in the monadic adjunction arising from the spherical adjunction

$$f^*\colon \mathcal{D}(k) \longleftrightarrow \mathrm{Fun}(S^n, \mathcal{D}(k)) : f_*\,,$$

where $\mathrm{Fun}(S^n, \mathcal{D}(k))$ is the ∞-category of $\mathcal{D}(k)$-valued local systems on the n-sphere S^n and f^* is the pullback along $f\colon S^n \to *$, see [Chr22b] for a detailed study of this adjunction. In the case $n = 2$, this adjunction describes the singularities of the perverse schober $\mathcal{F}_{\mathcal{T}}$ considered in Section 4.1. Let $M = f_* f^*$ be the adjunction monad and denote by $\mathrm{Fun}(S^n, \mathcal{D}(k))^{\mathrm{mnd}} := \mathrm{LMod}_M(\mathcal{D}(k))$ the stable, presentable ∞-category of left modules over M. We denote the associated fully faithful functor $\mathrm{Fun}(S^n, \mathcal{D}(k))^{\mathrm{mnd}} \to \mathrm{Fun}(S^n, \mathcal{D}(k))$ by i_{mnd} and we will identify $\mathrm{Fun}(S^n, \mathcal{D}(k))^{\mathrm{mnd}}$ with its essential image under i_{mnd}.

Lemma 2.8. *The object $f^*(k)$ is a compact generator of the ∞-category* $\mathrm{Fun}(S^n, \mathcal{D}(k))^{\mathrm{mnd}}$.

Proof. The restriction of f_* to $\mathrm{Fun}(S^n, \mathcal{D}(k))^{\mathrm{mnd}}$ is by definition monadic. Its left adjoint $(f^*)^{\mathrm{mnd}}$ is obtained from f^* by restricting the target. The k-linear functor $(f^*)^{\mathrm{mnd}}$ is fully determined by the image of k, which is $f^*(k)$. It is thus equivalent to the functor $(\text{-})\otimes f^*(k)$ (defined using the k-linear structure). The adjunction

$$(\text{-})\otimes f^*(k) \dashv \mathrm{Mor}_{\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}}(f^*(k),\text{-})\,,$$

and the facts that the right adjoint $\mathrm{Mor}_{\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}}(f^*(k),\text{-}) \simeq (f_*)^{\mathrm{mnd}}$ is conservative, because monadic, and that $k\in\mathcal{D}(k)$ is compact generator, now imply that $f^*(k)$ compactly generates $\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}$. □

As shown in [Chr22a, Prop. 5.5], there exists an equivalence of k-linear ∞-categories

$$\mathrm{Fun}(S^n,\mathcal{D}(k)) \simeq \mathcal{D}(k[t_{n-1}])\,,$$

where $k[t_{n-1}]$ denotes the graded polynomial algebra with generator in degree $|t_{n-1}| = n-1$.

Lemma 2.9. *There exist equivalences of k-linear ∞-categories*

$$\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}} \simeq \mathrm{Ind}\,\mathrm{Fun}(S^n,\mathcal{D}^{\mathrm{perf}}(k)) \simeq \mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])\,,$$

compatible with their inclusions into $\mathrm{Fun}(S^n,\mathcal{D}(k))$, where $\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$ denotes the derived ∞-category of $k[t_{n-1}]$-modules with finite dimensional homology.

Proof. Under the equivalence $\mathrm{Fun}(S^n,\mathcal{D}(k))\simeq\mathcal{D}(k[t_{n-1}])$, the forgetful functor $\mathcal{D}(k[t_{n-1}])\to\mathcal{D}(k)$ corresponds to the evaluation functor $\mathrm{Fun}(S^n,\mathcal{D}(k))\to\mathcal{D}(k)$ at any point $x\in S^n$. This implies that $\mathrm{Fun}(S^n,\mathcal{D}^{\mathrm{perf}}(k))\simeq\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$, which yields $\mathrm{Ind}\,\mathrm{Fun}(S^n,\mathcal{D}^{\mathrm{perf}}(k))\simeq\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$ by passing to Ind-completions.

We proceed by showing the equivalence $\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}\simeq\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$. Consider the trivial $k[t_{n-1}]$-module with homology k (corresponding to $f^*(k)$ under the equivalence $\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])\simeq\mathrm{Fun}(S^n,\mathcal{D}(k)^{\mathrm{perf}})$). Let $X\in\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$. We show by induction on the dimension of the homology of X, that X lies in the stable closure $\langle k\rangle\subset\mathcal{D}(k[t_{n-1}])$ of k, i.e. the smallest stable subcategory containing k. The induction beginning is the case $X\simeq 0$ and thus clear. For the induction step, suppose that $m\in\mathbb{Z}$ is the maximal degree in which the homology $\mathrm{H}_*\mathrm{Mor}_{\mathcal{D}(k[t_{n-1}])}(k[t_{n-1}],X)$ of X is nontrivial. We find a corresponding non-zero morphism $\alpha\colon k[t_{n-1}][m]\to X$, such that the composite with $k[t_{n-1}][m{+}n{-}1]\to k[t_{n-1}][m]$ is zero. The morphism α thus induces a non-zero morphism $\alpha'\colon k\to X$, which is injective on homology. Taking the fiber of α' we obtain a module $\mathrm{fib}(\alpha')$, whose homology has one dimension less. We have by the induction assumption, that $\mathrm{fib}(\alpha')\in\langle k\rangle$. From $X\simeq\mathrm{cof}(\mathrm{fib}(\alpha')\to k)$, it follows that X also lies in $\langle k\rangle$. Since any object in $\langle k\rangle$ also lies in $\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$, we obtain $\langle k\rangle=\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$. This ∞-category is stable and also idempotent-complete, as having finite dimensional homology is a condition preserved under retracts. Consider the k-linear endomorphism dg-algebras $\mathrm{End}_{\mathcal{D}(k[t_{n-1}])}(k)\simeq\mathrm{End}_{\mathrm{Fun}(S^n,\mathcal{D}(k))}(f^*(k))\simeq k\oplus k[-n]$. Since $f^*(k)$ is by Lemma 2.8 a compact generator of $\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}$, we have
$\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}\simeq\mathcal{D}(\mathrm{End}_{\mathrm{Fun}(S^n,\mathcal{D}(k))}(f^*(k)))\simeq\mathcal{D}(\mathrm{End}_{\mathcal{D}(k[t_{n-1}])}(k))$, see [Lur17, 7.1.2.1]. The perfect derived ∞-category $\mathcal{D}^{\mathrm{perf}}(\mathrm{End}_{\mathcal{D}(k[t_{n-1}])}(k))$ (consisting of compact objects) is by [Lur17, 7.2.4.1, 7.2.4.4] the smallest stable subcategory of $\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}\simeq\mathcal{D}(\mathrm{End}_{\mathcal{D}(k[t_{n-1}])}(k))$ containing k which is closed under retracts and thus equivalent to $\langle k\rangle=\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$. It follows that

$$\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}\simeq\mathcal{D}(\mathrm{End}_{\mathcal{D}(k[t_{n-1}])}(k))\simeq\mathrm{Ind}\,\mathcal{D}^{\mathrm{perf}}(\mathrm{End}_{\mathcal{D}(k[t_{n-1}])}(k))\simeq\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])\,,$$

concluding the proof. □

We denote by $k[t_{n-1}^{\pm}]$ the dg-algebra of graded Laurent polynomials with generator in degree $|t_{n-1}|=n-1$. Note that if $(n-1)$ is even, then $k[t_{n-1}]$ and $k[t_{n-1}^{\pm}]$ are graded commutative dg-algebras, whereas if $(n-1)$ is odd, then $k[t_{n-1}]$ and $k[t_{n-1}^{\pm}]$ are not graded commutative, because $t_{n-1}^2\neq 0$. By Lemma 2.6, the following Lemma shows that $\mathcal{D}(k[t_{n-1}^{\pm}])$ arises as the Verdier quotient of $\mathcal{D}(k[t_{n-1}])$ by $\mathrm{Ind}\,\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$, or by Lemma 2.9 equivalently as the Verdier quotient of $\mathrm{Fun}(S^n,\mathcal{D}(k))$ by $\mathrm{Fun}(S^n,\mathcal{D}(k))^{\mathrm{mnd}}$.

Lemma 2.10. *Let $n \geq 1$. The ∞-category $\mathcal{D}(k[t_{n-1}])$ admits a semiorthogonal decomposition $(\mathcal{D}(k[t_{n-1}^{\pm}]), \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}]))$.*

Proof. For the proof, we show that $\mathcal{D}(k[t_{n-1}^{\pm}]) = \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])^{\perp}$ is the stable subcategory of $\mathcal{D}(k[t_{n-1}])$ consisting of objects b, such that $\mathrm{Mor}_{\mathcal{D}(k[t_{n-1}])}(a,b) \simeq 0$ for all $a \in \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])$. Using that the inclusion $\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}]) \subset \mathcal{D}(k[t_{n-1}])$ admits a right adjoint, the Lemma then follows from [DKSS21, Prop. 2.2.4, Prop. 2.3.2].

The objects of $\mathcal{D}(k[t_{n-1}])$ can be identified with dg-modules over the dg-algebra $k[t_{n-1}]$. Using this identification, we observe that $\mathcal{D}(k[t_{n-1}^{\pm}]) \subset \mathcal{D}(k[t_{n-1}])$ is the full subcategory consisting of $k[t_{n-1}]$ dg-modules $M_\bullet$, such that $t_{n-1}\colon M_i \to M_{i+n-1}$ is an isomorphism of k-vector spaces for all $i \in \mathbb{Z}$. We have

$$\mathrm{Mor}_{\mathcal{D}(k[t_{n-1}])}(k[t_{n-1}][i], M_\bullet) \simeq M_{\bullet+i}\,.$$

Using that $k \simeq \mathrm{cof}(k[t_{n-1}][n-1] \to k[t_{n-1}])$, we thus have

$$\mathrm{Mor}_{\mathcal{D}(k[t_{n-1}])}(k[i], M_\bullet) \simeq \mathrm{cof}(t_{n-1}\colon M_{\bullet+i} \to M_{\bullet+i+n-1}) \in \mathcal{D}(k)\,.$$

We thus find that $t_{n-1}\colon M_i \to M_{i+n-1}$ is an isomorphism for all $i \in \mathbb{Z}$ if and only if $M_\bullet \in \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])^{\perp}$. This shows the desired equality

$$\mathcal{D}(k[t_{n-1}^{\pm}]) = \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_{n-1}])^{\perp}\,,$$

concluding the proof. □

Lemma 2.11. *There exists an equivalence of 1-categories $\mathrm{Vect}_k \simeq \operatorname{ho}\mathcal{D}(k[t_1^{\pm}])$, where Vect_k denotes the 1-category of k-vector spaces and $\operatorname{ho}\mathcal{D}(k[t_1^{\pm}])$ denotes the homotopy category of $\mathcal{D}(k[t_1^{\pm}])$.*

Proof. Since any complex of k-vector spaces is quasi-isomorphic to its homology, any 1-periodic complex with values in k is equivalent to an object in the image of $N(\mathrm{Vect}_k) \hookrightarrow \mathcal{D}(k) \xrightarrow{-\otimes k[t_1^{\pm}]} \mathcal{D}(k[t_1^{\pm}])$. This functor is thus essentially surjective, and using that

$$\pi_0 \mathrm{Map}_{\mathcal{D}(k[t_1^{\pm}])}(k[t_1^{\pm}], k[t_1^{\pm}]) \simeq \mathrm{H}_0(k[t_1^{\pm}]) \simeq k\,,$$

one sees that this functor is also fully faithful on the level of homotopy categories. □

In the triangulated homotopy category $\operatorname{ho}\mathcal{D}(k[t_1^{\pm}])$, distinguished triangles take the form

$$\ker(\alpha) \oplus \operatorname{coker}(\alpha) \xrightarrow{(\iota,0)} k[t_1^{\pm}]^{\oplus I} \xrightarrow{\alpha} k[t_1^{\pm}]^{\oplus J} \xrightarrow{(0,\pi)} \ker(\alpha) \oplus \operatorname{coker}(\alpha)$$

with ι being the kernel map and π being the cokernel map and I, J two sets.

Finally, we note an interpretation of the 1-periodic derived category in terms of the formal punctured neighborhood of infinity of [Efi17].

Proposition 2.12. *The category of perfect complexes on the formal punctured neighborhood of infinity of $\mathcal{D}^{\mathrm{perf}}(k[t_1])$ is equivalent to the perfect derived category of 1-periodic chain complexes $\mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}])$.*

Proof of Proposition 2.12. Consider the Verdier localization sequence

$$\mathcal{D}^{\mathrm{perf}}(k[\epsilon_{-2}]/\epsilon_{-2}^2) \longrightarrow \mathcal{D}^{\mathrm{perf}}(kQ) \longrightarrow \mathcal{D}^{\mathrm{perf}}(k[t_1])\,,$$

where $k[\epsilon_{-2}]/\epsilon_{-2}^2$ denotes the graded dual numbers with $|\epsilon_{-2}| = -2$ and kQ denotes the graded Kronecker quiver, with two vertices $1, 2$ and arrows $a, b\colon 1 \to 2$ in degrees $|a| = 0, |b| = 1$. The k-linear ∞-category $\mathcal{D}^{\mathrm{perf}}(kQ)$ is smooth and proper. The ∞-category of perfect complexes on the formal punctured neighborhood of infinity of $\mathcal{D}^{\mathrm{perf}}(k[t_1])$ is thus given by the (automatically idempotent complete) singularity category $\mathcal{D}^{\mathrm{fin}}(k[\epsilon_{-2}]/\epsilon_{-2}^2)/\mathcal{D}^{\mathrm{perf}}(k[\epsilon_{-2}]/\epsilon_{-2}^2)$. As shown for instance in the proof of Theorem 4.10 in [Ami09], this singularity category is equivalent to the the cosingularity category of $\mathcal{D}^{\mathrm{perf}}(k[t_1])$, which is by Lemma 2.10 given by

$$\mathcal{D}^{\mathrm{perf}}(k[t_1])/\mathcal{D}^{\mathrm{fin}}(k[t_1]) \simeq \mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}])\,,$$

as desired. □

2.4 Relative Calabi-Yau structures

Let k be a field and $\kappa = k$ or $\kappa = k[t_{2m}^{\pm}]$ the dg-algebra of graded Laurent polynomials, see Section 2.3, for some $m \in \mathbb{N}$. In this section, we sketch the definition of a relative right κ-linear Calabi-Yau structure. The notion of relative Calabi–Yau structure was described in case $\kappa = k$ the setting of dg-categories in [BD19] and generalized to arbitrary R-linear relative Calabi–Yau structures, with R an $\mathbb{E}_\infty$-ring spectrum, on stable ∞-categories in [Chr26].

Definition 2.13. Let $\mathcal{A}$ be a compactly generated κ-linear ∞-category. A Serre functor on $\mathcal{A}$ is a κ-linear endofunctor $U\colon \mathcal{A} \to \mathcal{A}$, satisfying that there exists a natural equivalence of functors

$$\mathrm{Mor}_{\mathcal{A}}(\text{-}_1, \text{-}_2) \simeq \mathrm{Mor}_{\mathcal{A}}(\text{-}_2, U(\text{-}_1))^*\colon \mathcal{A}^{\mathrm{c,op}} \times \mathcal{A}^{\mathrm{c}} \to \mathcal{D}(\kappa)\,.$$

Here $\mathcal{A}^{\mathrm{c}}$ denotes the full subcategory of $\mathcal{A}$ of compact objects and $(\text{-})^* = \mathrm{Mor}_{\mathcal{D}(\kappa)}(\text{-}, \kappa)\colon \mathcal{D}(\kappa)^{\mathrm{op}} \to \mathcal{D}(\kappa)$.

Every compactly generated, proper κ-linear ∞-category $\mathcal{A}$ admits a Serre functor U. This functor is obtained by identifying the bimodule right dual (i.e. the right adjoint) of the evaluation functor $\mathrm{Mor}_{\mathcal{D}}(\text{-},\text{-})\colon \mathcal{A}^\vee \otimes \mathcal{A} \to \mathcal{D}(\kappa)$ with an endofunctor $\mathrm{id}_{\mathcal{A}}^* = U$ of $\mathcal{A}$. If $\mathcal{A}$ is firthermore smooth, then the evaluation functor also admits a left dual, which we can identify with a functor $\mathrm{id}_{\mathcal{A}}^!$. In this case, the functor $\mathrm{id}_{\mathcal{A}}^!$ is inverse to $\mathrm{id}_{\mathcal{A}}^*$. We refer to [Chr26] for background. A weak right n-Calabi-Yau structure on such an ∞-category $\mathcal{A}$ consists of a natural equivalence $U \simeq \mathrm{id}_{\mathcal{A}}[n]$. Such an equivalence arises from a κ-linear dual Hochschild homology class

$$\sigma\colon \kappa[n] \to \mathrm{HH}(\mathcal{A})^* \simeq \mathrm{Mor}_{\mathrm{Lin}_\kappa(\mathcal{A},\mathcal{A})}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}}^*)\,,$$

where $\mathrm{Lin}_\kappa(\mathcal{A},\mathcal{A})$ denotes the κ-linear ∞-category of κ-linear endofunctors of $\mathcal{A}$. A right n-Calabi–Yau structure on $\mathcal{A}$ consists of a lift of a weak right Calabi–Yau structure σ to a dual cyclic homology class $\eta\colon \kappa[n] \to \mathrm{HH}(\mathcal{A})^*_{S^1}$.

Consider a κ-linear functor $G\colon \mathcal{B} \to \mathcal{A}$ between compactly generated, proper κ-linear ∞-categories that admits a κ-linear right adjoint H. The functor G induces a morphism $\mathrm{HH}(G)^*\colon\ \mathrm{HH}(\mathcal{A})^* \to \mathrm{HH}(B)^*$ on the dual Hochschild homology and we define $\mathrm{HH}(\mathcal{B},\mathcal{A})^* = \mathrm{cof}(\mathrm{HH}(G)^*)$. The relative dual cyclic homology $\mathrm{HH}(\mathcal{B},\mathcal{A})^*_{S^1}$ is defined similarly. A relative Hochschild class $\sigma\colon \kappa[n] \to \mathrm{HH}(\mathcal{B},\mathcal{A})^*$ gives rise to a diagram

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{B}} \arrow[r, "\mathrm{u}"] & HG \arrow[d] & \\
& H\,\mathrm{id}_{\mathcal{A}}^*\,G[1-n] \arrow[r, "\tilde{\mathrm{cu}}"] & \mathrm{id}_{\mathcal{B}}^*[1-n]
\end{tikzcd}$$

where u is the unit of $G \dashv H$ and $\tilde{\mathrm{cu}}$ a kind of counit, see [BD21, Chr26], together with a choice of null-homotopy of the composite morphism.

Definition 2.14. A right n-Calabi-Yau structure on the functor $G\colon \mathcal{B} \to \mathcal{A}$ consists of a class $\eta\colon \kappa[n] \to \mathrm{HH}(\mathcal{B},\mathcal{A})^*_{S^1}$, whose composite with $\mathrm{HH}(\mathcal{B},\mathcal{A})^*_{S^1} \to \mathrm{HH}(\mathcal{B},\mathcal{A})^*$ induces a diagram with horizontal fiber and cofiber sequences

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{B}} \arrow[r] \arrow[d] & HG \arrow[r] \arrow[d] & \mathrm{cof} \arrow[d] \\
\mathrm{fib} \arrow[r] & H\,\mathrm{id}_{\mathcal{A}}^*\,G[1-n] \arrow[r] & \mathrm{id}_{\mathcal{B}}^*[1-n]
\end{tikzcd} \tag{6}$$

whose vertical morphisms are equivalences.

Informally, relative right Calabi-Yau structures can be understood as follows. There is a morphism $U[-n] \simeq \mathrm{id}_{\mathcal{B}}^*[-n] \to \mathrm{id}_{\mathcal{B}}$ of endofunctors, with U the Serre functor, which is not an equivalence but whose fiber is equivalent to $HG[-1]$. Thus, a relative Calabi-Yau structure is a description of how much the Serre functor differs from a suspension of the identity, in terms of the adjunction $G \dashv H$.

3 Perverse schobers on surfaces

3.1 Marked surfaces and ribbon graphs

In this section, we recall standard notions regarding marked surfaces and their spanning ribbon graphs.

Definition 3.1. A surface is a smooth compact oriented surface $\mathbf{S}$ with non-empty boundary $\partial\mathbf{S}$ and interior $\mathbf{S}^\circ$. Note that this implies that the boundary of $\mathbf{S}$ is a disjoint union of circles.

A marked surface consists of a surface $\mathbf{S}$ and a finite set $M \subset \partial\mathbf{S}$ of marked points on the boundary of $\mathbf{S}$ such that each connected component of $\partial\mathbf{S}$ contains at least one marked point. We always assume that $\mathbf{S}$ is not the monogon or digon.

Definition 3.2.
1. A graph $\mathbf{G}$ consists of two finite sets $\mathbf{G}_0$ of vertices and H of halfedges together with an involution $\tau\colon \mathrm{H} \to \mathrm{H}$ and a map $\sigma\colon \mathrm{H} \to \mathbf{G}_0$.
2. Let $\mathbf{G}$ be a graph. We denote the set of orbits of τ by $\mathbf{G}_1$. The elements of $\mathbf{G}_1$ are called the edges of $\mathbf{G}$. An edge is called internal if the orbit contains two elements and called external if the orbit contains a single element. An internal edge is called a loop at $v \in \mathbf{G}_0$ if it consists of two halfedges both being mapped under σ to v. The set of internal edges of $\mathbf{G}$ is denoted by $\mathbf{G}_1^\circ$ and the set of external edges is denoted by $\mathbf{G}_1^\partial$.
3. A ribbon graph consists of a graph $\mathbf{G}$ together with a choice of a cyclic order on the set $\mathrm{H}(v)$ of halfedges incident to v for each $v \in \mathbf{G}_0$.

Definition 3.3. Let $\mathbf{G}$ be a graph. We denote by $\mathrm{Exit}(\mathbf{G})$ the simplicial set given by the nerve of the 1-category with

- objects given by the set $\mathbf{G}_0 \amalg \mathbf{G}_1$ and
- non-identity morphisms of the form $v \to e$ with $v \in \mathbf{G}_0$ a vertex and $e \in \mathbf{G}_1$ an edge incident to v. If e is a loop at v, then there are two morphisms $v \to e$.

We call $\mathrm{Exit}(\mathbf{G})$ the exit path category of $\mathbf{G}$.

The geometric realization $|\mathbf{G}|$ of the graph $\mathbf{G}$ is defined as the geometric realization $|\mathrm{Exit}(\mathbf{G})|$ of the simplicial set $\mathrm{Exit}(\mathbf{G})$.

Remark 3.4. Let $\mathbf{G}$ be a graph and $\mathbf{S}$ an oriented surface. An embedding of $|\mathbf{G}|$ into $\mathbf{S}$ determines a canonical ribbon graph structure on $\mathbf{G}$ by requiring the halfedges at any vertex to be ordered in the counterclockwise direction.

Definition 3.5. Let $\mathbf{S}$ be a marked surface. We call a graph $\mathbf{G}$ together with an embedding $i\colon |\mathbf{G}| \subset \mathbf{S}\backslash M$ a spanning graph for $\mathbf{S}$, if i is a homotopy equivalence and the restriction $i^{-1}(\partial\mathbf{S}\backslash M) \to \partial\mathbf{S}\backslash M$ of i is also a homotopy equivalence. We consider spanning graphs as endowed with the canonical ribbon graph structure arising from the embedding into the oriented surface $\mathbf{S}$.

Remark 3.6. Any trivalent spanning graph of a marked surface $\mathbf{S}$ arises as the dual graph of an ideal triangulation of $\mathbf{S}$ in the sense of Definition 6.5 and this defines a bijection between sets of suitable homotopy classes of trivalent spanning graphs and ideal triangulations.

3.2 Perverse schobers on marked surfaces

In this section, we recall the definition of a perverse schober parametrized by a ribbon graph. The notion of a perverse schober was first proposed by Kapranov–Schechtman in [KS14], though they only give a definition for perverse schobers on a disc. The notion of a perverse schober on a ribbon graph spanning a surface was introduced in [Chr22a, Sections 3 and 4].

We first briefly recall the definition of spherical functor or a spherical adjunction. Let $F\colon \mathcal{A} \leftrightarrow \mathcal{B} : G$ be an adjunction between stable ∞-categories with unit $\mathrm{u}\colon \mathrm{id}_{\mathcal{A}} \to GF$ and counit $\mathrm{cu}\colon FG \to \mathrm{id}_{\mathcal{B}}$. Taking the cofiber of the unit u in the stable ∞-category $\mathrm{Fun}(\mathcal{A},\mathcal{A})$ of endofunctors, we obtain a functor $T_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ called the twist functor. Similarly, the cotwist functor $T_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}$ is defines as the fiber of the counit. The adjunction $F \dashv G$ is called spherical if both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ are autoequivalences. In this case, we also call the functors

F and G spherical. The notion of a spherical functor was first proposed for dg categories in [AL17]. For treatments of spherical adjunctions in the setting of stable ∞-categories, we refer to [DKSS21, Chr22b].

We turn to describing the local model of a perverse schober arising from a spherical functor described in [Chr22a, Section 3.1], based on the relative $S_\bullet$-construction of [Dyc21]. Let $\mathbf{G}$ be a ribbon graph and v a vertex of $\mathbf{G}$ of valency m. The objects of the under category $\mathrm{Exit}(\mathbf{G})_{v/}$ are the vertex v and its incident halfedges. We choose a labeling of these halfedges by $1,\dots,m$, compatibly with their given cyclic order. Given a spherical functor $F_v\colon \mathcal{V}_v \to \mathcal{N}$, we define a functor $\mathcal{G}_v(F_v)\colon \mathrm{Exit}(\mathbf{G})_{/v} \to \mathrm{St}$ as follows. Consider the diagram $D\colon [m-1] \to \mathrm{Set}_\Delta$ given by

$$\mathcal{V}_v \xrightarrow{F_v} \underbrace{\mathcal{N} \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \mathcal{N}}_{(m-1)\text{-many}}\,. \tag{7}$$

Let $p\colon \Gamma(D) \to \Delta^{m-1} \simeq N([m-1])$ be the (covariant) Grothendieck construction of D, see [Lur09, Def. 3.2.5.2], where it is called the relative nerve. The Grothendieck construction is an explicit model for the coCartesian fibration classified by the functor D.

- We set $\mathcal{G}_v(F_v)(v) = \mathrm{Fun}_{\Delta^{m-1}}(\Delta^{m-1}, \Gamma(D))$ to be the stable ∞-category of sections of p. We also denote $\mathcal{V}^m_{F_v} = \mathcal{G}_v(F_v)(v)$. We note that if $\mathcal{V}_v \simeq 0$, then $\mathcal{V}^m_{F_v} \simeq \mathrm{Fun}(\Delta^{m-2}, \mathcal{N})$, this should be considered as the ∞-category of representations of the A_{n-1}-quiver in $\mathcal{N}$. In general, $\mathcal{V}^m_{F_v}$ is a model for the lax limit of the diagram (7) in the $(\infty,2)$-category of stable ∞-categories.
- We set $\mathcal{G}_v(F_v)(i) = \mathcal{N}$ for all $1,\dots,m$.
- We denote $\varrho_i \coloneqq \mathcal{G}_v(F_v)(v \to i)\colon \mathcal{V}^m_{F_v} \to \mathcal{N}$. We set

  $$\varrho_1 = \mathrm{ev}_{m-1}$$

  to be the evaluation functor of the sections of p at $m-1 \in \Delta^{m-1}$. For $1 \le i < m$, we set $\varrho_{i+1} = \mathrm{ladj}(\mathrm{ladj}(\varrho_i))$. For a detailed description (and thus also a proof of existence) of these functors, see [Chr22a, Lem. 3.3].

In the case $m = 1$, we have $\mathrm{Exit}(\mathbf{G})_{/v} \simeq \Delta^1$ and the functor $\mathscr{G}_v(F_v)$ is simply the functor F_v. One can further show that making a different choice of total order on the halfedges of v changes the functor $\mathcal{G}_v(F_v)$ only by composition with an autoequivalence, see [Chr22a, Prop. 3.11].

Definition 3.7 ([Chr22a, Def. 4.14])**.** Let $\mathbf{G}$ be a ribbon graph. A $\mathbf{G}$-parametrized perverse schober is a functor $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathcal{P}r^L_{\mathrm{St}}$ such that for each vertex v of $\mathbf{G}$ there exists a spherical functor $F_v\colon \mathcal{V}_v \to \mathcal{N}$, satisfying that the composite of $\mathrm{Exit}(\mathbf{G})_{v/} \to \mathrm{Exit}(\mathbf{G})$ with $\mathcal{F}$ is equivalent to $\mathcal{G}_v(F_v)$ in $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G})_{v/}\ \mathrm{St})$.

Given a vertex v of $\mathbf{G}$, the ∞-category $\mathcal{V}_v$ is called the ∞-category of vanishing cycles at v. If $\mathcal{V}_v \not\simeq 0$, we call v a singularity of $\mathcal{F}$. We call F_v the spherical functor describing or underlying $\mathcal{F}$ at v (this functor is only determined up to equivalence and composition with autoequivalences).

Assuming that the graph $\mathbf{G}$ is connected, a $\mathbf{G}$-parametrized perverse schober assigns to each edge of $\mathbf{G}$ an equivalent ∞-category $\mathcal{N}$, called the generic stalk of the perverse schober, or also the ∞-category of nearby cycles.

Remark 3.8. We restrict in Definition 3.7 to perverse schobers taking values in stable and presentable ∞-categories and colimit preserving functors, but one could also consider perverse schobers valued in small stable ∞-categories.

Definition 3.9. Let $\mathbf{G}$ be a ribbon graph and $\mathcal{F}$ a $\mathbf{G}$-parametrized perverse schober.

(1) The ∞-category of global sections $\mathcal{H}(\mathbf{G},\mathcal{F}) \in \mathcal{P}r^L_{\mathrm{St}}$ of $\mathcal{F}$ is defined as the limit of $\mathcal{F}$.

(2) The ∞-category of lax sections $\mathcal{L}(\mathbf{G},\mathcal{F}) = \mathrm{Fun}_{\mathrm{Exit}(\mathbf{G})}(\mathrm{Exit}(\mathbf{G}), \Gamma(\mathcal{F})) \in \mathcal{P}r^L_{\mathrm{St}}$ of $\mathcal{F}$ is defined as the ∞-category of sections of the Grothendieck construction $p\colon \Gamma(\mathcal{F}) \to \mathrm{Exit}(\mathbf{G})$.

Remark 3.10. As a model for the limit $\mathcal{H}(\mathbf{G},\mathcal{F})$ of a $\mathbf{G}$-parametrized perverse schober $\mathcal{F}$, we usually consider the full subcategory of the ∞-category of sections $\mathcal{L}(\mathbf{G},\mathcal{F})$ of the Grothendieck construction $p\colon \Gamma(\mathcal{F}) \to \operatorname{Exit}(\mathbf{G})$ of $\mathcal{F}$ spanned by coCartesian sections. This description of the limit is proven in [Lur, Tag 03AB]. A section $s\colon \operatorname{Exit}(\mathbf{G}) \to \Gamma(\mathcal{F})$ of p is called coCartesian if $s(\alpha)$ is a coCartesian edge in the dual sense of [Lur09, 2.4.1.1] for each 1-simplex α in $\operatorname{Exit}(\mathbf{G})$.

Let $\mathcal{F}$ be a $\mathbf{G}$-parametrized perverse schober. Given an edge e of $\mathbf{G}$, we denote by $\mathrm{ev}_e\colon \mathcal{H}(\mathbf{G},\mathcal{F}) \to \mathcal{F}(e)$ the evaluation functor, which maps a coCartesian section s of p to its value $s(e)$ at e. Consider the product of the evaluation functors at the external edges

$$\prod_{e\in \mathbf{G}_1^\partial} \mathrm{ev}_e\colon \mathcal{H}(\mathbf{G},\mathcal{F}) \to \prod_{e\in \mathbf{G}_1^\partial} \mathcal{F}(e)\,.$$

The functor $\prod_{e\in \mathbf{G}_1^\partial} \mathrm{ev}_e$ preserves all limits and colimits by [Lur09, 5.1.2.3, 4.3.1.10, 4.3.1.16]. By the ∞-categorical adjoint functor theorem, $\prod_{e\in \mathbf{G}_1^\partial} \mathrm{ev}_e$ thus admits a left adjoint.

Definition 3.11. The functor

$$\partial\mathcal{F}\colon \prod_{e\in \mathbf{G}_1^\partial} \mathcal{F}(e) \longrightarrow \mathcal{H}(\mathbf{G},\mathcal{F})$$

is defined as the left adjoint of $\prod_{e\in \mathbf{G}_1^\partial} \mathrm{ev}_e$.

Remark 3.12. In the study of partially wrapped Fukaya categories, particularly Fukaya-Seidel categories, the functor $\partial\mathcal{F}$ is often called the Orlov functor or cup functor and $\prod_{e\in \mathbf{G}_1^\partial} \mathrm{ev}_e$ is called the cap functor, see for instance [Syl19] for background.

The functor $\partial\mathcal{F}$ is always spherical:

Theorem 3.13 ([Chr25a, Cor. 4.7])**.** *The adjunction*

$$\partial\mathcal{F}\colon \prod_{e\in \mathbf{G}_1^\partial} \mathcal{F}(e) \longleftrightarrow \mathcal{H}(\mathbf{G},\mathcal{F}) : \prod_{e\in \mathbf{G}_1^\partial} \mathrm{ev}_e$$

is spherical.

3.3 Fiber and cofiber sequences of perverse schobers

In this section, we introduce a novel notion of a fiber and cofiber sequence of parametrized perverse schobers. For this, we first give a definition of the ∞-category of parametrized perverse schobers.

Definition 3.14. Let $\mathcal{F},\mathcal{G}$ be $\mathbf{G}$-parametrized perverse schobers (which we assumed to taking values in $\mathcal{P}r^L_{\mathrm{St}}$). We call a natural transformation $\eta\colon \mathcal{F}\to\mathcal{G}$ in $\operatorname{Fun}(\operatorname{Exit}(\mathbf{G}),\mathcal{P}r^L_{\mathrm{St}})$ a morphism of perverse schobers if η is locally right adjointable, by which we mean that for every morphism $v \xrightarrow{h} e$ in $\operatorname{Exit}(\mathbf{G})$ with corresponding diagram

$$\begin{tikzcd} \mathcal{F}(v) \arrow[r, "\eta_v"] \arrow[d, "\mathcal{F}(v\xrightarrow{h}e)"'] & \mathcal{G}(v) \arrow[d, "\mathcal{G}(v\xrightarrow{h}e)"] \\ \mathcal{F}(e) \arrow[r, "\eta_e"] & \mathcal{G}(e) \end{tikzcd} \tag{8}$$

the mate

$$\begin{aligned} \eta_v \circ \mathrm{radj}(\mathcal{F}(v\to e)) &\xrightarrow{\text{unit}} \mathrm{radj}(\mathcal{G}(v\to e))\circ \mathcal{G}(v\to e)\circ \eta_v \circ \mathrm{radj}(\mathcal{F}(v\to e)) \\ &\simeq \mathrm{radj}(\mathcal{G}(v\to e))\circ \eta_e \circ \mathcal{F}(v\to e)\circ \mathrm{radj}(\mathcal{F}(v\to e)) \\ &\xrightarrow{\text{counit}} \mathrm{radj}(\mathcal{G}(v\to e))\circ \eta_e \end{aligned}$$

is an equivalence.

Definition 3.15. The ∞-category of $\mathbf{G}$-parametrized perverse schobers $\mathfrak{P}(\mathbf{G})$ is defined as the subcategory of $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}), \mathcal{P}r^L_{\mathrm{St}})$ consisting of $\mathbf{G}$-parametrized perverse schobers and morphisms of such in the sense of Definition 3.14.

The adjointability condition on the square (8) may be called vertical right adjointability. There are three further similar conditions, called horizontal/vertical left/right adjointability. Such adjointability conditions are also called *Beck–Chevalley* conditions.

If the functors in (8) admit left adjoints, then

- right horizontal adjointability is equivalent to left vertical adjointability and
- right vertical adjointability is equivalent to left horizontal adjointability.

Furthermore, the next Proposition 3.16 shows that the two right adjointability conditions are also equivalent.

Proposition 3.16 ([CDW23, Prop. 4.5.6])**.** *Consider a commutative diagram of stable ∞-categories*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "G"] \arrow[d, "F_{\mathcal{V}}"] & \mathcal{N} \arrow[d, "F_{\mathcal{N}}"] \\ \mathcal{V}' \arrow[r, "G'"] & \mathcal{N}' \end{tikzcd}$$

where G and G' are spherical functors. Suppose that $F_{\mathcal{V}}$ and $F_{\mathcal{N}}$ admit right adjoints $F_{\mathcal{V}}^R$ and $F_{\mathcal{N}}^R$. Then the square is horizontally right adjointable if and only if it is vertically right adjointable.

Proof. This follows directly from the proof of [CDW23, Prop. 4.5.6]. □

Definition 3.17. An inclusion of $\mathbf{G}$-parametrized perverse schobers consists of a morphism of perverse schobers $\iota\colon \mathcal{F} \to \mathcal{G}$, such that $\iota(x)\colon \mathcal{F}(x) \to \mathcal{G}(x)$ is fully faithful for all $x \in \mathrm{Exit}(\mathbf{G})$.

Definition 3.18. A fiber and cofiber sequence of $\mathbf{G}$-parametrized perverse schobers consists of $\mathbf{G}$-parametrized perverse schobers $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}$, together with morphisms of perverse schobers $\iota\colon \mathcal{F}_2 \to \mathcal{G}$ and $\pi\colon \mathcal{G} \to \mathcal{F}_1$, such that the square

$$\begin{tikzcd} \mathcal{F}_2 \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{G} \arrow[d, "\pi"] \\ 0 \arrow[r] & \mathcal{F}_1 \end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}), \mathcal{P}r^L_{\mathrm{St}})$ commutes and is pullback and pushout.

We note that if the square in Definition 3.18 is pullback and pushout, then ι is an inclusion of perverse schobers, and the square is also pullback and pushout in $\mathfrak{P}(\mathbf{G})$. Since we are in the presentable setting, this can also be considered a semiorthogonal decomposition of perverse schobers, see Lemma 2.6. Indeed, it gives rise to semiorthogonal decompositions on the local and global levels:

Lemma 3.19. *Let $\mathcal{F}_2 \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{F}_1$ be a fiber and cofiber sequence of $\mathbf{G}$-parametrized perverse schobers.*

(1) For each $x \in \mathrm{Exit}(\mathbf{G})$, the ∞-category $\mathcal{G}(x)$ inherits a semiorthogonal decomposition $(\mathcal{F}_1(x), \mathcal{F}_2(x))$.

(2) The ∞-category of global sections $\mathcal{H}(\mathbf{G}, \mathcal{G})$ inherits a semiorthogonal decomposition $(\mathcal{H}(\mathbf{G}, \mathcal{F}_1), \mathcal{H}(\mathbf{G}, \mathcal{F}_2))$.

Proof. Using that evaluation at $x \in \mathrm{Exit}(\mathbf{G})$ defines a limit and colimit preserving functor $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}), \mathcal{P}r^L_{\mathrm{St}}) \to \mathcal{P}r^L_{\mathrm{St}}$, part (1) follows from Lemma 2.6.

The functor $\mathcal{F}_2$ is by assumption the fiber of π in $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}), \mathcal{P}r^L_{\mathrm{St}})$. Since limits commute with limits, we find that there exists a fiber sequence in $\mathcal{P}r^L_{\mathrm{St}}$

$$\mathcal{H}(\mathbf{G}, \mathcal{F}_2) \xrightarrow{i} \mathcal{H}(\mathbf{G}, \mathcal{G}) \xrightarrow{\pi} \mathcal{H}(\mathbf{G}, \mathcal{F}_1)\,.$$

Passing to right adjoints yields a cofiber sequence in $\mathcal{P}^R_{\mathrm{St}}$, showing by Lemma 2.6 that $\mathcal{H}(\mathbf{G}, \mathcal{G})$ inherits the desired semiorthogonal decomposition $\big(\mathcal{H}(\mathbf{G}, \mathcal{F}_1), \mathcal{H}(\mathbf{G}, \mathcal{F}_2)\big)$. $\square$

Definition 3.20. Let $\mathcal{F}$ be a $\mathbf{G}$-parametrized perverse schober. Recall that given a vertex v of $\mathbf{G}$, $\mathcal{F}$ is described near v by a spherical functor $F\colon \mathcal{V}_v \to \mathcal{N}$, where $\mathcal{V}_v$ is called the ∞-category of vanishing cycles and $\mathcal{N}$ the ∞-category of nearby cycles. We call

- $\mathcal{F}$ vanishing-monadic, if at each vertex v of $\mathbf{G}$ the spherical functor $F_v\colon \mathcal{V}_v \to \mathcal{N}$ describing $\mathcal{F}$ at v is a monadic functor.
- $\mathcal{F}$ nearby-monadic, if at each vertex v of $\mathbf{G}$ the right adjoint of the spherical functor F_v describing $\mathcal{F}$ at v is a monadic functor.
- $\mathcal{F}$ locally constant if $\mathcal{F}$ has no singularities.

Remark 3.21. A spherical functor between presentable ∞-categories is monadic if and only if it is comonadic if and only if it is conservative, as follows from the Lurie–Barr–Beck theorem [Lur17, Thm. 4.7.3.5].

Finally, we show that the cofiber morphism of a morphism of perverse schobers is automatically a morphism of perverse schobers.

Lemma 3.22. *Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathbf{G}$-parametrized perverse schober. Let $\pi\colon \mathcal{G} \to \mathcal{F}'$ be the cofiber morphism in $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}), \mathcal{P}r^L_{\mathrm{St}})$, meaning it appears in the pushout diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G} \arrow[d, "\pi"] \\
0 \arrow[r] & \mathcal{F}'
\end{tikzcd}$$

where 0 denotes the constant perverse schober with value 0. If $\mathcal{F}'$ is a $\mathbf{G}$-parametrized perverse schober, then π defines a morphism of perverse schobers.

Proof. Let $v \in \mathbf{G}_0$ be a vertex with an incident halfedge $h \in e \in \mathbf{G}_1$. Consider the following diagram of presentable ∞-categories

$$\begin{tikzcd}
\mathcal{F}(v) \arrow[r, shift left, "f_v"] \arrow[d, "\mathcal{F}(h)"'] & \mathcal{G}(v) \arrow[l, shift left, "f_v^R"] \arrow[r, shift left, "\pi_v"] \arrow[d, "\mathcal{G}(h)"] & \mathcal{F}'(v) \arrow[l, shift left, "\pi_v^R"] \arrow[d, "\mathcal{F}'(h)"] \\
\mathcal{F}(e) \arrow[r, shift left, "\iota_e"] & \mathcal{G}(e) \arrow[l, shift left, "f_e^R"] \arrow[r, shift left, "\pi_e"] & \mathcal{F}'(e) \arrow[l, shift left, "\pi_e^R"]
\end{tikzcd}$$

where the horizontal rightpointed morphisms define cofiber sequences and the superscript R denotes right adjoints. The left square commutes in both directions, by the right adjointability of the square. The right square commutes in the horizontal right direction and by Proposition 3.16 it suffices to show that it is horizontally right adjointable.

The image of the fully faithful functor π_v^R is given by the fiber of f_v^R. Therefore, the image of $\mathcal{G}(h) \circ \pi_v^R$ lies in the fiber of f_e^R, and thus factors uniquely through the inclusion $\pi_e^R\colon \mathcal{F}'(e) \hookrightarrow \mathcal{G}(e)$ via a functor $F\colon \mathcal{F}'(v) \to \mathcal{F}'(e)$. The equivalences

$$\begin{aligned}
F &\simeq \pi_e \circ \pi_e^R \circ F \\
&\simeq \pi_e \circ \mathcal{G}(h) \circ \pi_v^R \\
&\simeq \mathcal{F}'(h) \circ \pi_v \circ \pi_v^R \\
&\simeq \mathcal{F}'(h)
\end{aligned}$$

thus show that the right square commutes in the right vertical direction. We use this to prove that it is also right adjointable:

We must show that the natural transformation

$$\begin{aligned}\mathcal{G}(h) \circ \pi_v^R &\to \pi_e^R \circ \pi_e \circ \mathcal{G}(h) \circ \pi_v^R \\ &\simeq \pi_e^R \circ \mathcal{F}'(h) \circ \pi_v \circ \pi_v^R \\ &\to \pi_e^R \circ \mathcal{F}'(h)\end{aligned}$$

is a natural equivalence. The third natural transformation above, given by the counit of $\pi_v \dashv \pi_v^R$, is an equivalence by the fully faithfulness of π_v^R. Using that $\mathcal{G}(h) \circ \pi_v^R \simeq \pi_e^R \circ \mathcal{F}'(h)$, we see that the first natural transformation applies the unit of $\pi_e \dashv \pi_e^R$ to a functor with image contained in the image of π_e^R. Restricted to the image of π_e^R, the unit is an equivalence. $\square$

4 Cosingularity categories and perverse schobers

We fix a field k and a marked surface $\mathbf{S}$ with a trivalent spanning graph $\mathcal{T}$. Theorem 6.1 in [Chr22a] constructs a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$, whose ∞-category of global sections $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ is equivalent to the unbounded derived ∞-category of the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ associated with $\mathcal{T}$. The potential of the underlying quiver consists of 3-cycles, one for each triangle of the dual ideal triangulation of $\mathcal{T}$. In Sections 4.1 and 4.2, we describe the cosingularity category associated to $\mathscr{G}_{\mathcal{T}}$ in terms of the global sections of a quotient perverse schober of $\mathcal{F}_{\mathcal{T}}$. Using this, we are able to deduce in Section 4.3 that the cosingularity category is equivalent to the 1-periodic topological Fukaya category.

4.1 A fiber and cofiber sequence of perverse schobers

Consider the $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$ from [Chr22a, Theorem 6.1]. The spherical adjunction underlying $\mathcal{F}_{\mathcal{T}}$ at any vertex of $\mathcal{T}$ is given by

$$f^*\colon \mathcal{D}(k) \longleftrightarrow \operatorname{Fun}(S^2, \mathcal{D}(k)) : f_*$$

where f^* is the pullback functor along $S^2 \to *$. There is a k-linear equivalence $\operatorname{Fun}(S^2, \mathcal{D}(k)) \simeq \mathcal{D}(k[t_1])$, where $k[t_1]$ is the graded polynomial algebra with $|t_1| = 1$, under which the functor f^* is identified with the pullback functor ϕ^* along $\phi\colon k[t_1] \xrightarrow{t_1 \mapsto 0} k$, see [Chr22a, Prop. 5.5]. In particular, the functor f^* is conservative and the adjunction $f^* \dashv f_*$ thus comonadic. The adjunction $f^* \dashv f_*$ is however not monadic. The Eilenberg-Moore ∞-category $\operatorname{Fun}(S^2, \mathcal{D}(k))^{\mathrm{mnd}}$ of the monad $f_* f^*$ can be identified with the full subcategory $\operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k)) \subset \operatorname{Fun}(S^2, \mathcal{D}(k))$, see Lemma 2.9. Using this observation, we can obtain a nearby-monadic and vanishing-monadic $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$ by restricting in the construction of $\mathcal{F}_{\mathcal{T}}$ in [Chr22a] the spherical adjunction $f^* \dashv f_*$ to the spherical monadic and comonadic adjunction

$$(f^*)^{\text{Ind-fin}}\colon \mathcal{D}(k) \longleftrightarrow \operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k)) : f_*^{\text{Ind-fin}}\,.$$

We remark that to give this definition of $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$, we also use that in the construction of $\mathcal{F}_{\mathcal{T}}$ all appearing autoequivalences of $\operatorname{Fun}(S^2, \mathcal{D}(k))$ preserve the subcategory $\operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k)) \subset \operatorname{Fun}(S^2, \mathcal{D}(k))$.

There is an apparent pointwise inclusion $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}}$ in $\operatorname{Fun}(\operatorname{Exit}(\mathcal{T}, \mathcal{P}r^L_{\mathrm{St}})$.

Lemma 4.1. *The inclusion $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}}$ is a morphism of perverse schobers.*

Proof. By Proposition 3.16, it suffices to show to show that for $1 \leq i \leq 3$, the diagram

$$\begin{tikzcd} \mathcal{V}^3_{(f^*)^{\text{Ind-fin}}} \arrow[r, hook, "i_{\mathcal{V}}"] \arrow[d, "\varrho_i"'] & \mathcal{V}^3_{f^*} \arrow[d, "\varrho_i"] \\ \operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[r, hook, "i_{\mathrm{mnd}}"] & \operatorname{Fun}(S^2, \mathcal{D}(k)) \end{tikzcd} \tag{9}$$

is horizontally right adjointable. We only consider the case $i = 1$, the other cases can be treated analogously or also follow using [Chr22a, Prop. 3.11] from the $i = 1$ case.

It is straightforward to see that the following diagram commutes,

$$\begin{tikzcd}
\mathcal{D}(k) \arrow[r, "\mathrm{id}"] \arrow[d, "(f^*)^{\text{Ind-fin}}"'] & \mathcal{D}(k) \arrow[r, "\mathrm{id}"] \arrow[d, "f^*"] & \mathcal{D}(k) \arrow[d, "(f^*)^{\text{Ind-fin}}"] \\
\operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[r, "i_{\mathrm{mnd}}"] \arrow[d, "\mathrm{id}"'] & \operatorname{Fun}(S^2, \mathcal{D}(k)) \arrow[r, "\pi_{\mathrm{mnd}}"] \arrow[d, "\mathrm{id}"] & \operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[d, "\mathrm{id}"] \\
\operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[r, "i_{\mathrm{mnd}}"] & \operatorname{Fun}(S^2, \mathcal{D}(k)) \arrow[r, "\pi_{\mathrm{mnd}}"] & \operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}^{\mathrm{perf}}(k))
\end{tikzcd}$$

where π_{mnd} denotes the right adjoint of i_{mnd}. The right adjoint of the functor $i_{\mathcal{V}}$ between lax limits is obtained by passing componentwise to the right adjoint, see [CDW23, Remark A.2.5], and thus acts on the third component via π_{mnd}. This shows the desired right adjointability. □

Definition 4.2. We define the functor $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}} \colon \operatorname{Exit}(\mathcal{T}) \to \mathcal{P}r^L_{\mathrm{St}}$ as the cofiber of the inclusion $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \hookrightarrow \mathcal{F}_{\mathcal{T}}$ in $\operatorname{Fun}(\operatorname{Exit}(\mathcal{T}), \mathcal{P}r^L_{\mathrm{St}})$.

Lemma 4.3. *The functor $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ is a $\mathcal{T}$-parametrized perverse schober. At each vertex of $\mathcal{T}$, it is described by the trivial spherical adjunction*

$$0 \leftrightarrow \mathcal{D}(k[t_1^{\pm}])\,.$$

Proof. Using Lemma 2.10, this is straightforward to check by using that pushouts in the functor category $\operatorname{Fun}(\operatorname{Exit}(\mathcal{T}), \mathrm{St})$ are computed pointwise in $\operatorname{Exit}(\mathcal{T})$. □

Remark 4.4. The abbreviation clst stands for 'cluster', or rearranging the letters to lcst, it stands for 'locally constant'. Note that the perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ is locally constant in the sense of Definition 3.20, meaning that $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ has no singularities.

Proposition 4.5. *The given morphisms $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ define a fiber and cofiber sequence of $\mathcal{T}$-parametrized perverse schobers.*

Proof. The inclusion $\alpha \colon \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}}$ is by Lemma 4.1 a morphism of perverse schobers. Lemma 3.22 shows that the cofiber morphism $\mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ arising from the definition of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ is also a morphism of perverse schobers. Thus $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ indeed forms a fiber and cofiber sequence of perverse schobers. □

4.2 The cosingularity category of a marked surface

Definition 4.6. Let $\mathcal{D}$ be a compactly generated k-linear ∞-category.

(1) We define the finite subcategory $\mathcal{D}^{\mathrm{fin}} \subset \mathcal{D}$ as the full subcategory consisting of objects $Y \in \mathcal{D}$, satisfying that $\operatorname{Mor}_{\mathcal{D}}(X, Y) \in \mathcal{D}(k)$ is perfect for all compact objects $X \in \mathcal{D}^c$. We note if $\mathcal{D}$ admits a compact generator Z, then Y is finite if and only if $\operatorname{Mor}_{\mathcal{D}}(Z, Y)$ is perfect.

(2) Suppose that $\mathcal{D}$ is smooth. Then the evaluation functor of $\mathcal{D}$ admits a left adjoint, which we identify with an endofunctor $\mathrm{id}^!_{\mathcal{D}}$ of $\mathcal{D}$ (also known as the inverse dualizing bimodule). Suppose further that $\mathrm{id}^!_{\mathcal{D}}$ is an equivalence. Then the finite objects in $\mathcal{D}$ are compact, i.e. $\mathcal{D}^{\mathrm{fin}} \subset \mathcal{D}^c$, see [Chr26, Lem. 2.25]. Passing to Ind-completions yields the fully faithful inclusion $\operatorname{Ind}\mathcal{D}^{\mathrm{fin}} \subset \mathcal{D}$. We define the cosingularity category of $\mathcal{D}$ as the Ind-complete Verdier quotient $\mathcal{D}/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}$ (i.e. cofiber in $\mathcal{P}r^L_{\mathrm{St}}$).

Remark 4.7. Let $\mathcal{D}$ be as in part (2) of Definition 4.6. Then $\mathcal{D}$ admits a semiorthogonal decomposition $(\mathcal{D}/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}, \operatorname{Ind}\mathcal{D}^{\mathrm{fin}})$ into its cosingularity category and its Ind-finite part, see Lemma 2.6.

The goal of this section is to prove the following theorem.

Theorem 4.8. *The cosingularity category of $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$ is equivalent to the ∞-category $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ of global sections of the perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$.*

Remark 4.9. A shown in [Chr22a, Section 6.4], the ∞-category $\mathcal{D}(\mathscr{G}_{\mathcal{T}}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ does not depend on the choice of trivalent spanning graph $\mathcal{T}$ of the marked surface $\mathbf{S}$, up to equivalence of ∞-categories. The same thus also holds for its associated (Ind-complete) cosingularity category. We therefore may write

$$\mathcal{C}_{\mathbf{S}} \coloneqq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})\,.$$

Note that we will not distinguish in notation between the Ind-complete cosingularity category and $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$.

To prove Theorem 4.8, we show that the ∞-category of global sections of $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$ is equivalent to $\operatorname{Ind}\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\mathrm{fin}}$ and then make use of the fiber and cofiber sequence $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$. The argument for the former relies on a computational argument using compact generators arising from curves described in [Chr21].

Definition 4.10. We denote by $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ the full subcategory of global sections $X \in \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ satisfying that $\mathrm{ev}_e(X) \in \operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}(k)^{\mathrm{perf}})$ for all edges e of $\mathcal{T}$.

Lemma 4.11. *The ∞-category $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$ is compactly generated by the objects M_γ^L associated to finite matching data $(\gamma, f^*(k))$ with γ pure and $f^*(k) \in \operatorname{Fun}(S^2, \mathcal{D}(k))^{\mathrm{fin}} \subset \operatorname{Fun}(S^2, \mathcal{D}(k))$, as defined in [Chr21].*

Proof. Using that $f^*(k)$ is a compact generator of $\operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}(k))^{\mathrm{fin}}$, [Chr21, Prop. 4.20] shows that a certain collection of matching data with local value $f^*(k)$ and pure, finite underlying matching curves compactly generate $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$. We remark that for the fact that the matching curves giving rise to this compact generator are finite, it is crucial that the marked surface has a marked point on each boundary component. □

Lemma 4.12. *For e an edge of $\mathcal{T}$, consider the left adjoint*

$$\mathrm{ev}_e^L \colon \mathcal{D}(k[t_1]) \simeq \mathcal{F}_{\mathcal{T}}(e) \to \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$$

of the evaluation functor ev_e at e. The object $\bigoplus_{e\in\mathcal{T}_1} \mathrm{ev}_e^L(k[t_1]) \in \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ is identified under the equivalence $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}) \simeq \mathcal{D}(\mathscr{G}_{\mathcal{T}})$ with the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}} \in \mathcal{D}(\mathscr{G}_{\mathcal{T}})$. In particular, $\bigoplus_{e\in\mathcal{T}_1} \mathrm{ev}_e^L(k[t_1])$ is a compact generator.

Proof. This is shown in [Chr22a, Prop. 6.7]. □

Proposition 4.13. *There exist equivalences of ∞-categories*

$$\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}} \simeq \operatorname{Ind}\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}}\,.$$

Proof. It follows from the definition of $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$, that the image on global sections of the inclusion $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}}$ consists of global sections which evaluate on each edge of $\mathcal{T}$ to an object in $\operatorname{Ind}\operatorname{Fun}(S^2, \mathcal{D}(k)^{\mathrm{perf}}) \subset \operatorname{Fun}(S^2, \mathcal{D}(k))$. We thus see that $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$.

We proceed with showing the equivalence $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}} \simeq \operatorname{Ind}\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}}$. We can realize both of these ∞-categories as presentable, stable subcategories of $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$. An object of $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ is by Lemma 4.12 finite if and only if its evaluations to the edges of $\mathcal{T}$ are finite in $\operatorname{Fun}(S^2, \mathcal{D}(k))$, which is equivalent to lying in $\operatorname{Fun}(S^2, \mathcal{D}(k)^{\mathrm{perf}}) \subset \operatorname{Fun}(S^2, \mathcal{D}(k))$. It follows that $\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$. Using that the objects in $\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}}$ are compact in $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$, we further get $\operatorname{Ind}\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$. Lemma 4.11, in combination with the fact that the global sections associated to finite, pure matching data with finite local value, in the sense of [Chr21], lie by construction in $\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}} \subset \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$, now implies that both $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})^{\text{Ind-fin}}$ and $\operatorname{Ind}\mathcal{D}(\mathscr{G}_{\mathcal{T}})^{\mathrm{fin}}$ are compactly generated by the same set of objects and thus equivalent. □

Proof of Theorem 4.8. Combine Proposition 4.5, Lemma 3.19 and Proposition 4.13. □

The dg-algebra $k[t_2^{\pm}]$ is commutative and the ∞-category $\mathcal{D}(k[t_1^{\pm}])$ comes with an apparent $k[t_2^{\pm}]$-linear structure. This can be used to construct a factorization of the perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ through $\mathrm{LinCat}_{k[t_2^{\pm}]} \to \mathrm{St}$. In particular, its limit $\mathcal{C}_{\mathbf{S}} = \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ inherits a $k[t_2^{\pm}]$-linear structure.

Proposition 4.14. *The $k[t_2^{\pm}]$-linear ∞-category $\mathcal{C}_{\mathbf{S}}$ is smooth and proper.*

Proof. Note that $\mathcal{C}_{\mathbf{S}}$ is equivalent to the colimit of the right adjoint diagram of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ in $\mathrm{LinCat}_{k[t_2^{\pm}]}$. The smoothness thus follows from the fact that finite colimits of smooth $k[t_2^{\pm}]$-linear ∞-categories along compact objects preserving functors are again smooth, see [Chr26, Cor. 3.12].

To see that $\mathcal{C}_{\mathbf{S}}$ is proper, consider the compact generator $\bigoplus_{e\in\mathcal{T}_1} \mathrm{ev}_e^L(k[t_1^{\pm}])$ of $\mathcal{C}_{\mathbf{S}}$, given by the sum of the images of the left adjoints of the evaluation functors $\mathrm{ev}_e\colon \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}) \to \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e) \simeq \mathcal{D}(k[t_1^{\pm}])$. Proposition 4.20 in [Chr21] shows that $\mathrm{ev}_e^L(k[t_1^{\pm}]) \simeq M_{c_e}^{k[t_1^{\pm}]}$ for some matching datum $(c_e, k[t_1^{\pm}])$, where c_e is a finite, pure matching curve (or equivalently $\mathrm{ev}_e^L(k[t_1^{\pm}]) \simeq M_{c_e}$ in the notation of Section 6.2 below for some matching curve c_e). Using Theorems 6.8 and 6.9, we see that $\mathrm{End}(\bigoplus_{e\in\mathcal{T}_1} \mathrm{ev}_e^L(k[t_1^{\pm}]))$ is perfect in $\mathcal{D}(k[t_2^{\pm}])$, showing that $\mathcal{C}_{\mathbf{S}}$ is proper. □

4.3 Relation to the 1-periodic topological Fukaya category

The perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ is locally constant, i.e. has no singularities, and its generic stalk is the derived ∞-category $\mathcal{D}(k[t_1^{\pm}])$ of 1-periodic chain complexes. It is thus locally described by the 1-periodic derived category of the A_2-quiver. Global sections of locally constant perverse schobers may be referred to as topological Fukaya categories, as these recover in special cases the constructions given in [DK18, DK15, HKK17]. We refer to the ∞-category of global sections $\mathcal{C}_{\mathbf{S}} = \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ as the topological Fukaya category of $\mathbf{S}$ with values in the derived ∞-category of 1-periodic chain complexes, or the 1-periodic topological Fukaya category for short.

There is a notion of monodromy for parametrized perverse schobers [Chr26] and this monodromy of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ along any closed curve in $\mathbf{S}^\circ$ is trivial, this is shown in the proof of Theorem 5.5 in [Chr21]. Furthermore, any locally constant parametrized perverse schober is fully determined up to equivalence by its generic stalk and monodromy, see [Chr26, Prop. 4.34]. We are hence justified in calling the cosingularity category $\mathcal{C}_{\mathbf{S}}$ *the* 1-periodic topological Fukaya category of $\mathbf{S}$.

The ∞-category $\mathcal{C}_{\mathbf{S}}^{c}$ can further be considered as an ∞-categorical avatar of the topological Fukaya (dg-)category in the sense of [DK18] with coefficients in the cyclic 2-Segal object arising from the Waldhausen $S_\bullet$-construction of the 2-periodic dg-category $\mathrm{dgMod}_{k[t_1^{\pm}]}$ of dg $k[t_1^{\pm}]$-modules. In particular, Corollary 3.4.7 of [DK18] shows the following.

Theorem 4.15. *The stable ∞-category $\mathcal{C}_{\mathbf{S}}$ is acted upon by automorphisms in $\mathrm{ho}\,\mathrm{LinCat}_k$ by the mapping class group of the surface $\mathbf{S}$ of isotopy classes of orientation preserving diffeomorphisms $\mathbf{S} \to \mathbf{S}$ restricting to the identity on $\partial\mathbf{S}$.*

Finally, we also record a description of the cosingularity category $\mathcal{C}_{\mathbf{S}}$ in terms of a gentle algebra. A description of $\mathcal{C}_{\mathbf{S}}$ as the orbit ∞-category $\mathcal{D}(\mathrm{gtl})/[1]$ is given in [Chr25b, Prop. 4.11].

Proposition 4.16. *There exists an ungraded gentle algebra* gtl *and an equivalence of k-linear ∞-categories $\mathcal{C}_{\mathbf{S}} \simeq \mathcal{D}(\mathrm{gtl}) \otimes_k \mathcal{D}(k[t_1^{\pm}])$.*

Proof. Choose a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$. As observed in [HKK17], the ∞-category of global sections of a locally constant perverse schober with generic stalk $\mathcal{D}(k)$ on $\mathbf{S}$ parametrized by $\mathcal{T}$ admits a formal generator whose endomorphism algebra is a finite dimensional, and in general graded, gentle algebra, denoted gtl'. Using that tensoring with $\mathcal{D}(k[t_1^{\pm}])$ (with respect to the symmetric monoidal structure of LinCat_k) preserves colimits in LinCat_k, we find an equivalence $\mathcal{C}_{\mathbf{S}} \simeq \mathcal{D}(\mathrm{gtl}') \otimes_k \mathcal{D}(k[t_1^{\pm}])$ in LinCat_k. Let gtl be the ungraded gentle algebra obtained from gtl' by discarding the grading. It is easy to see that $\mathcal{D}(\mathrm{gtl}') \otimes_k \mathcal{D}(k[t_1^{\pm}]) \simeq \mathcal{D}(\mathrm{gtl}) \otimes_k \mathcal{D}(k[t_1^{\pm}])$ in LinCat_k, showing the claim. $\square$

Topological Fukaya categories admit relative Calabi–Yau structures if the monodromy of the perverse schober acts trivially on a relative Calabi–Yau structure of the generic stalk, see [Chr26, Thm. 5.8]. In our setting, this gives the following.

Theorem 4.17 ([Chr26, Thm. 6.5])**.** *Assume that* $\mathrm{char}(k) \neq 2$. *The* $k[t_2^{\pm}]$*-linear functor*

$$\prod_{e \in \mathcal{T}_1^{\partial}} \mathrm{ev}_e \colon \mathcal{C}_{\mathbf{S}} \longrightarrow \prod_{e \in \mathcal{T}_1^{\partial}} \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e)$$

admits a 2*-Calabi-Yau structure in the sense of Definition 2.14.*

5 Exact structures from relative Calabi-Yau structures

Consider an exact functor $G \colon \mathcal{B} \to \mathcal{A}$ between stable ∞-categories, meaning a functor which preserves finite limits and colimits. We can pull back the split-exact structure on $\mathcal{A}$ to an exact structure on the ∞-category $\mathcal{B}$, such that a sequence in $\mathcal{B}$ is exact if and only if its image under G is a split-exact sequence in $\mathcal{A}$, see Example 5.8. The goal of this section is to show that if G carries a right 2-Calabi-Yau structure and is spherical, then the exact structure on $\mathcal{B}$ is Frobenius and its extriangulated homotopy category is 2-Calabi-Yau.

In Section 5.1, we recall the notion of an exact ∞-category and describe pullbacks of exact structures. In Section 5.2, we recall what is an extriangulated category and why it is natural to consider the extriangulated homotopy categories of exact ∞-categories. In Section 5.3, we prove that the Frobenius extriangulated structure arising from a functor carrying a right 2-Calabi-Yau structure is extriangulated 2-Calabi–Yau.

5.1 Exact ∞-categories

Definition 5.1 ([Bar15])**.** An exact ∞-category is a triple $(\mathcal{C}, \mathcal{C}_{\dagger}, \mathcal{C}^{\dagger})$, where $\mathcal{C}$ is an additive ∞-category and $\mathcal{C}_{\dagger}, \mathcal{C}^{\dagger} \subset \mathcal{C}$ are subcategories (called subcategories of inflations and deflations), satisfying that

(1) every morphism $0 \to X$ in $\mathcal{C}$ lies in $\mathcal{C}_{\dagger}$ and every morphism $X \to 0$ in $\mathcal{C}$ lies in $\mathcal{C}^{\dagger}$.

(2) pushouts in $\mathcal{C}$ along morphisms in $\mathcal{C}_{\dagger}$ exist and lie in $\mathcal{C}_{\dagger}$. Dually, pullbacks in $\mathcal{C}$ along morphisms in $\mathcal{C}^{\dagger}$ exist and lie in $\mathcal{C}^{\dagger}$.

(3) Given a commutative square in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"] & Y \arrow[d, "c"] \\ X' \arrow[r, "d"] & Y' \end{tikzcd}$$

the following are equivalent.

- The square is pullback, $c \in \mathcal{C}_{\dagger}$ and $d \in \mathcal{C}^{\dagger}$.
- The square is pushout, $b \in \mathcal{C}_{\dagger}$ and $a \in \mathcal{C}^{\dagger}$.

When the exact structure is clear from the context, we also simply refer to $\mathcal{C}$ as an exact ∞-category.

Definition 5.2. An exact sequence $X \to Y \to Z$ in an exact ∞-category $(\mathcal{C}, \mathcal{C}_{\dagger}, \mathcal{C}^{\dagger})$ consists of a fiber and cofiber sequence in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "b"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

with $a \in \mathcal{C}_{\dagger}$ and $b \in \mathcal{C}^{\dagger}$.

Example 5.3. 1. Let $\mathcal{C}$ be an additive ∞-category. Then there is an exact ∞-category $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ with inflations consisting of inclusions of direct summands and deflations consisting of projections onto direct summands, called the split-exact structure. The exact sequences are the split-exact sequences $X \hookrightarrow X \oplus Y \twoheadrightarrow Y$.

2. Let $\mathcal{C}$ be a stable ∞-category. Then $(\mathcal{C}, \mathcal{C}, \mathcal{C})$ is an exact ∞-category.

Definition 5.4. Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ be an exact ∞-category.

1) An object $P \in \mathcal{C}$ is called projective if every exact sequence $X \to Y \to P$ is split-exact. An object $I \in \mathcal{C}$ is called injective if every exact sequence $I \to Y \to Z$ is split-exact.

2) We say that $\mathcal{C}$ has enough projectives if for each object $X \in \mathcal{C}$ there exists an exact sequence $Y \to P \to X$ with P projective. Similarly, we say that $\mathcal{C}$ has enough injectives if for each object $X \in \mathcal{C}$ there exists an exact sequence $X \to I \to Y$ with I injective.

3) We say that $\mathcal{C}$ is Frobenius if $\mathcal{C}$ has enough projectives and injectives and the classes of projective and injective objects coincide.

The ∞-categorical version of the stable 1-category of a Frobenius exact 1-category is the following. We thank Jasso–Kvamme–Palu–Walde for communicating this result.

Proposition 5.5. *Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ be a Frobenius exact ∞-category and W the class of morphisms $f\colon X \to Y$ which fit into an exact sequence*

$$X \xrightarrow{(f,g)} Y \oplus I \longrightarrow J$$

with I and J injective and g arbitrary. Then, the ∞-categorical localisation $\bar{\mathcal{C}} := \mathcal{C}[W^{-1}]$ of $\mathcal{C}$ at W is a stable ∞-category.

Proof sketch. Applying [Cis19, Prop. 7.5.6] and its dual version, one finds that $\bar{\mathcal{C}}$ admits finite limits and colimits and pushouts and pullbacks coincide. $\square$

Definition 5.6. 1. An exact functor $G\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ between exact ∞-categories consists of a functor $G\colon \mathcal{C} \to \mathcal{D}$ which preserves zero objects, inflations, deflations, as well as exact sequences[1].

2. A sub-exact structure of an exact ∞-category $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ consists of an exact ∞-category $(\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger)$, satisfying that $\mathcal{C}_\ddagger \subset \mathcal{C}_\dagger$ and $\mathcal{C}^\ddagger \subset \mathcal{C}^\dagger$.

Sub-exact structures can be pulled back along exact functors as follows.

Lemma 5.7. *Let $G\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be an exact functor between exact ∞-categories and $(\mathcal{D}, \mathcal{D}_\ddagger, \mathcal{D}^\ddagger)$ a sub-exact structure of $(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$. Then there exists a sub-exact structure $(\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger)$ of $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, such that G defines an exact functor $G\colon (\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger) \to (\mathcal{D}, \mathcal{D}_\ddagger, \mathcal{D}^\ddagger)$.*

Proof. We set $\mathcal{C}_\ddagger \subset \mathcal{C}_\dagger$ to be the subcategory of morphisms whose image under G lies in $\mathcal{D}_\ddagger$. We define $\mathcal{C}^\ddagger$ similarly. It is straightforward to verify that $(\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger)$ is an exact ∞-category and that $G\colon (\mathcal{C}, \mathcal{C}_\ddagger, \mathcal{C}^\ddagger) \to (\mathcal{D}, \mathcal{D}_\ddagger, \mathcal{D}^\ddagger)$ is an exact functor. $\square$

Example 5.8. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories and $G\colon \mathcal{C} \to \mathcal{D}$ is an exact functor in the usual sense, i.e. a functor that preserves finite limits and colimits. Then G defines an exact functor $G\colon (\mathcal{C}, \mathcal{C}, \mathcal{C}) \to (\mathcal{D}, \mathcal{D}, \mathcal{D})$ between exact ∞-categories. Applying Lemma 5.7, we obtain an additional exact structure $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ on $\mathcal{C}$ by pulling back the split-exact structure $(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ on $\mathcal{D}$. A fiber and cofiber sequence in $\mathcal{C}$ is exact in $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ if and only if its image under G is split-exact.

Proposition 5.9. *Let $G\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories. If G is a spherical functor, then the exact ∞-category $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ from Example 5.8 is Frobenius. The subcategory of injective and projective objects of $\mathcal{C}$ consists of the additive closure of the essential image of the right adjoint H of G.*

[1]This is equivalent to the a priori slightly stronger assumption of preserving pushouts along inflations and pullbacks along deflations.

Remark 5.10. A result similar to Proposition 5.9 appears for triangulated categories in [BS21], see Theorem 4.23 and Remark 4.24 in loc. cit.

Proof of Proposition 5.9. Let F be the left adjoint of G and H the right adjoint of G. By the sphericalness of G, we have that $H \simeq F \circ T_{\mathcal{D}}$, with $T_{\mathcal{D}}$ the cotwist functor of the adjunction $G \dashv H$, see [DKSS21, Cor. 2.5.16]. In particular, it follows that the essential images of F and H, and hence also their additive closures in $\mathcal{C}$, agree.

We begin by showing that every object of the form $F(d) \in \mathcal{C}$ with $d \in \mathcal{D}$ is injective. A dual argument shows that every object of the form $H(d) \in \mathcal{C}$ is projective. Let $c \in \mathcal{C}$ and consider an extension $\alpha \in \mathrm{Ext}^1_{\mathcal{C}}(F(d), c)$. The extension α corresponds to a fiber and cofiber sequence $c \to z \to F(d)$ in $\mathcal{C}$. The image of this fiber and cofiber sequence under G splits (i.e. defines an exact sequence) if and only if $G(\alpha) \simeq 0$. The commutative diagram,

$$\begin{tikzcd} & \mathrm{Ext}^1_{\mathcal{D}}(GF(d), G(c)) \arrow[dr, "\mathrm{u}"] & \\ \mathrm{Ext}^1_{\mathcal{C}}(F(d), c) \arrow[ur, "G"] \arrow[rr, "\simeq"'] & & \mathrm{Ext}^1_{\mathcal{D}}(d, G(c)) \end{tikzcd}$$

with u the unit of $F \dashv G$, shows that $G(\alpha) \simeq 0$ if and only if $\alpha \simeq 0$. This is the case, if and only if the fiber and cofiber sequence $c \to z \to F(d)$ already splits. Since c and α were chosen arbitrarily, it follows that $F(d)$ is injective.

Next, we show that $\mathcal{C}$ has enough injective and enough projective objects. Let $c \in \mathcal{C}$. We have a fiber and cofiber sequence $c \xrightarrow{\mathrm{u}_c} HG(c) \to T_{\mathcal{C}}(c)$, where u_c is a unit map and $T_{\mathcal{C}}$ the twist functor of $G \dashv H$. We apply G and extend to the diagram

$$\begin{tikzcd} & T_{\mathcal{D}}G(c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ G(c) \arrow[r, "G\,\mathrm{u}_c"'] \arrow[d] \arrow[dr, phantom, "\square"] & GHG(c) \arrow[r, "\mathrm{cu}_{G(c)}"'] \arrow[d] \arrow[dr, phantom, "\square"] & G(c) \arrow[d] \\ 0 \arrow[r] & GT_{\mathcal{C}}(c) \arrow[r] & 0 \end{tikzcd}$$

with $\mathrm{cu}_{G(c)}$ a counit map and all squares biCartesian. This shows that the sequence

$$G(c) \xrightarrow{G\,\mathrm{u}_c} GHG(c) \to GT_{\mathcal{C}}(c)$$

in $\mathcal{D}$ splits and hence that $c \xrightarrow{\mathrm{u}_c} HG(c) \to T_{\mathcal{C}}(c)$ is exact. We note that $HG(x)$ is both projective and injective, and have thus shown that $\mathcal{C}$ has enough injective objects. Replacing c with $T_{\mathcal{C}}^{-1}(c)$ in the above argument, we obtain the exact sequence $T_{\mathcal{C}}^{-1}(c) \xrightarrow{\mathrm{u}_{T_{\mathcal{C}}^{-1}(c)}} HGT_{\mathcal{C}}^{-1}(x) \to c$, showing that $\mathcal{C}$ also has enough projectives.

Finally, we show that any injective or projective object lies in the additive closure (closure under taking direct sums of direct summands) of the essential image of H. Let I be injective. Then we have an exact sequence $I \to HG(I) \to T_{\mathcal{C}}(I)$ in $\mathcal{C}$. Since I is injective, this sequence splits, showing that I is a direct summand of $HG(I)$, as desired. If P is projective, we similarly find an exact sequence $T_{\mathcal{C}}^{-1}(P) \to FG(P) \to P$, showing that P is a direct summand of $FG(P) \simeq HT_{\mathcal{D}}^{-1}G(P)$, as desired, concluding the proof. □

5.2 Extriangulated categories

Extriangulated categories were introduced by Nakaoka-Palu in [NP19] as a simultaneous generalization of triangulated and exact 1-categories. An extriangulated category $(C, \mathbb{E}, \mathfrak{s})$ consists of

- an additive 1-category C,
- an additive bifunctor $\mathbb{E}\colon C^{\mathrm{op}} \times C \to \mathrm{Ab}$ to the additive 1-category of abelian groups, of which we think as describing an interesting class of extensions in C and

- a sequence $(Y \to Z \to X) = \mathfrak{s}(\alpha)$, called realization, associated to each $\alpha \in \mathbb{E}(X, Y)$,

subject to number of conditions, see [NP19]. We will refer to the realization sequences as exact sequences in C, though Nakaoka-Palu call them conflation sequences.

Examples of extriangulated categories are, besides triangulated and exact categories, extension closed subcategories of triangulated categories. A further natural source of extriangulated categories are the homotopy 1-categories of exact ∞-categories. To see this, note that by [Kle22] any exact ∞-category admits a stable hull, meaning that it can be embedded as an extension closed subcategory in a stable ∞-category. Passing to homotopy categories, one obtains an extriangulated structure on the homotopy category, as it is an extension closed subcategory of the triangulated homotopy category of the stable hull. An independent and more direct proof that the homotopy category is extriangulated also appears in [NP20].

In the remainder of this article, we will assume all extriangulated categories to be k-linear, with k a field, meaning that the 1-category C is k-linear and $\mathbb{E}$ factors through $\mathrm{Vect}_k \to \mathrm{Ab}$.

Given a k-linear exact ∞-category $\mathcal{C}$, the additive functor of extensions $\mathbb{E}\colon \operatorname{ho}\mathcal{C}^{\mathrm{op}} \times \operatorname{ho}\mathcal{C} \to \mathrm{Vect}_k$ in the extriangulated structure of $\operatorname{ho}\mathcal{C}$ describes isomorphism classes of exact sequences in $\mathcal{C}$. The functor $\mathbb{E}$ can be useful for studying the exact ∞-category $\mathcal{C}$. For instance, one can use it to formulate a notion of cluster tilting object in the exact ∞-category $\mathcal{C}$, and this is simply an object which becomes cluster tilting in the extriangulated homotopy category $\operatorname{ho}\mathcal{C}$ in the sense of Definition 5.15 below.

The definition of a Frobenius extriangulated category is analogous to the definition of a Frobenius exact ∞-category.

Definition 5.11. Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangualated category.

1) An object $P \in C$ is called projective if $\mathbb{E}(P, X) \simeq 0$ for all $X \in C$ and injective if $\mathbb{E}(X, P) \simeq 0$ for all $X \in C$.
2) We say that C has enough projectives if for each object $X \in C$ there exists an exact sequence $Y \to P \to X$ with P projective. Similarly, we say that C has enough injectives if for each object $X \in C$ there exists an exact sequence $X \to I \to Y$ with I injective.
3) We say that C is Frobenius if C has enough projectives and injectives and the classes of projective and injective objects coincide.

The inflations, deflations and exact sequences in an exact ∞-category can be read off from the extriangulated structure of its homotopy category. Similarly, the condition of being a projective or injective object in an exact ∞-category can be tested in the extriangulated homotopy category. We also get the following.

Lemma 5.12. *A (k-linear) exact ∞-category is Frobenius if and only if its extriangulated homotopy category is Frobenius.*

Remark 5.13. Given a Frobenius extriangulated category C, its stable category $\bar{C}$ is defined as the quotient category by the ideal of morphisms which factor through an injective object. The stable category $\bar{C}$ inherits the structure of a triangulated category, see Corollary 7.4 and Remark 7.5 in [NP19].

If $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is a (k-linear) Frobenius exact ∞-category, the triangulated homotopy category of its associated stable ∞-category $\bar{\mathcal{C}}$ can be identified with the triangulated stable category $\overline{\operatorname{ho}\mathcal{C}}$ of the Frobenius extriangulated homotopy category $\operatorname{ho}\mathcal{C}$.

Definition 5.14. An extriangulated category $(C, \mathbb{E}, \mathfrak{s})$ is called extriangulated 2-Calabi-Yau if there exists an isomorphism of vector spaces $\mathbb{E}(X, Y) \simeq \mathbb{E}(Y, X)^*$, bifunctorial in X and Y.

We conclude this section with the definition of a cluster tilting object in an extriangulated category.

Definition 5.15. Let T be an object in an extriangulated category $(C, \mathbb{E}, \mathfrak{s})$.

- We denote by $\mathrm{add}(T) \subset C$ the additive closure of T, given by the subcategory given by objects equivalent to finite direct sums of direct summands of T.

- T is called basic if it can be decomposed into finitely many indecomposable direct summands which are pairwise non-isomorphic.
- T is called rigid if
$$\mathbb{E}(T,T) \simeq 0\,.$$
Further, a basic rigid object T is called maximal rigid, if there exist no object $X \in C$, satisfying that $T \oplus X$ is rigid and $X \notin \operatorname{add}(T)$.
- T is called cluster tilting if it is a basic rigid object and for every $X \in C$ there exists an exact sequence $T_1 \to T_0 \to X$ with $T_0, T_1 \in \operatorname{add}(T)$.

Lemma 5.16. *Let $(C, \mathbb{E}, \mathfrak{s})$ be a Frobenius extriangulated category with finite-dimensional Homs and $T \in C$ a basic rigid object.*

(1) Then T is a cluster tilting object if and only if for every object $X \in C$ the condition $\mathbb{E}(X,T) \simeq 0$ implies that $X \in \operatorname{add}(T)$.

(2) Dually, the following two statements are equivalent:
- *For every $X \in C$ there exists an exact sequence $X \to T_0 \to T_1$ with $T_0, T_1 \in \operatorname{add}(T)$.*
- *For every $X \in C$, the condition $\mathbb{E}(T,X) \simeq 0$ implies that $X \in \operatorname{add}(T)$.*

Note that if C is extriangulated 2-Calabi–Yau, this shows that T is a cluster tilting object if and only if the equivalent conditions of (2) hold.

Proof. The proof of part (2) is analogous to the proof of part (1), we thus only prove part (1).

Assume that T is a cluster tilting object. Choose an exact sequence $T_1 \to T_0 \to X$ in C with $T_0, T_1 \in \operatorname{add}(T)$. The exact sequence of vector spaces, see [GNP21, Thm. 3.5],
$$\operatorname{Hom}(T_0,T) \to \operatorname{Hom}(T_1,T) \to \mathbb{E}(X,T) \to \underbrace{\mathbb{E}(T_0,T)}_{\simeq 0}$$
shows that if $\mathbb{E}(X,T) \simeq 0$, then the sequence $T_1 \to T_0 \to X$ splits (since the connecting homomorphism vanishes), so that X is a direct summand of T_0.

For the converse direction, assume that for all $X \in C$, the condition $\mathbb{E}(X,T) \simeq 0$ implies that $X \in \operatorname{add}(T)$. Since C is Frobenius, we can find a deflation $P \to X$ with P projective. We define $T_0 = P \oplus (T \otimes \operatorname{Hom}_C(T,X)) \in \operatorname{add}(T)$. The apparent morphisms $T_0 \to X$ is again a deflation by [NP19, Cor. 3.16]. It remains to show that the term T_1 appearing in the exact sequence $T_1 \to T_0 \to X$ lies in $\operatorname{add}(T)$. We have an exact sequence of vector spaces
$$\operatorname{Hom}(T,T_0) \to \operatorname{Hom}(T,X) \to \mathbb{E}(T,T_1) \to \underbrace{\mathbb{E}(T,T_0)}_{\simeq 0}$$
see again [GNP21, Thm. 3.5]. The first morphisms is surjective by the definition of T_0, which shows that $\mathbb{E}(T,T_1) \simeq 0$. It follows that $T_1 \in \operatorname{add}(T)$, concluding the proof. $\square$

5.3 Exact structures via relative Calabi-Yau structures

Let $\kappa = k$ be a field or $\kappa = k[t_{2m}^{\pm}]$ the commutative dg-algebra of graded Laurent polynomials over a field. We fix a functor $G\colon \mathcal{B} \to \mathcal{A}$ between κ-linear smooth and proper ∞-categories, which is spherical and carries a right n-Calabi-Yau structure for some $n \in \mathbb{Z}$.

We begin this section by exhibiting a relative version of the triangulated n-Calabi-Yau property $\operatorname{Ext}^i_{\mathcal{B}}(X,Y) \simeq \operatorname{Ext}^{n-i}_{\mathcal{B}}(Y,X)^*$ using the relative right Calabi-Yau structure of $\mathcal{B}$. In the case $n = 2$, this relative version of duality amounts to a 2-Calabi–Yau property on the Frobenius extriangulated structure on $\operatorname{ho}\mathcal{B}$, see Proposition 5.21.

Definition 5.17. Let $X, Y \in \mathcal{B}^c$ be compact objects. We denote by $\operatorname{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y) \subset \operatorname{Ext}^i_{\mathcal{B}}(X,Y) \in N(\mathrm{Vect}_k)$ the kernel of the morphism
$$G\colon \operatorname{Ext}^i_{\mathcal{B}}(X,Y) \longrightarrow \operatorname{Ext}^i_{\mathcal{A}}(G(X),G(Y))\,. \tag{10}$$
We call $\operatorname{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y)$ the k-vector space of *Calabi-Yau extensions*.

Lemma 5.18. *The Calabi-Yau extensions assemble into a subfunctor*

$$\mathrm{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(\text{-},\text{-}) \subset \mathrm{Ext}^{i}_{\mathcal{B}}(\text{-},\text{-})\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \to \mathrm{N}(\mathrm{Vect}_k)\,.$$

Proof. Given morphisms $f\colon W \to X$, $g\colon X \to Y[i]$ and $h\colon Y[i] \to Z$ in B^{c} such that $G(g) = 0$, we have $G(hgf) \simeq G(h) \circ G(g) \circ G(f) \simeq 0$. It follows that Calabi–Yau extensions are closed under composition with morphisms and thus define a subfunctor. □

Given a k-vector space B, we write $B^* = \mathrm{Hom}_{\mathrm{Vect}_k}(B,k)$.

Proposition 5.19. *Let $X, Y \in \mathcal{B}^{\mathrm{c}}$ be compact objects. There exists an equivalence in $N(\mathrm{Vect}_k)$*

$$\mathrm{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y) \simeq \mathrm{Ext}^{n-i,\mathrm{CY}}_{\mathcal{B}}(Y,X)^*\,,$$

bifunctorial in X, Y.

We give the proof of Proposition 5.19 further below.

Let $(\mathcal{B}, \mathcal{B}_\dagger, \mathcal{B}^\dagger)$ be the exact ∞-category obtained by pulling back the split-exact structure on $\mathcal{A}$ along G, see Example 5.8. We denote by $(\mathrm{ho}\,\mathcal{B}^{\mathrm{c}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}, \mathfrak{s})$ the arising extriangulated homotopy category. Here, we use that $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}$ describes the extensions in this extriangulated category, as follows from Lemma 5.20. To be precise, we also abuse notation by labeling by $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}$ the second functor in the factorization

$$\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}\colon \mathcal{B}^{\mathrm{c,op}} \times \mathcal{B}^{\mathrm{c}} \longrightarrow \mathrm{ho}\,\mathcal{B}^{\mathrm{c,op}} \times \mathrm{ho}\,\mathcal{B}^{\mathrm{c}} \longrightarrow N(\mathrm{Vect}_k)\,.$$

Lemma 5.20. *Let $X, Y \in \mathcal{B}^{\mathrm{c}}$. Consider a fiber and cofiber sequence $X \to Z \xrightarrow{\alpha} Y$ in $\mathcal{B}$ and let $\beta\colon Y \to X[1]$ be the cofiber morphism of α. Then β lies in $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}(Y,X) \subset \mathrm{Ext}^{1}_{\mathcal{B}}(Y,X)$ if and only if the image of the fiber and cofiber sequence under G splits.*

Proof. We show that the fiber and cofiber sequence $G(X) \to G(Z) \xrightarrow{G(\alpha)} G(Y)$ splits if and only if the cofiber morphism $G(\beta)$ vanishes. By definition, the latter is equivalent to β being a Calabi-Yau extension. The forward implication is clear. For the converse, suppose that $G(\beta)$ is zero. Then its fiber morphism, given by $G(\alpha)$, is equivalent to $G(X) \oplus G(Y) \xrightarrow{(0,\mathrm{id})} G(Y)$. This shows that the fiber and cofiber sequence splits. □

Proposition 5.21. *Suppose that $n = 2$. The extriangulated category $(\mathrm{ho}\,\mathcal{B}^{\mathrm{c}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{B}}, \mathfrak{s})$ is Frobenius and extriangled 2-Calabi-Yau. The exact ∞-category $(\mathcal{B}^{\mathrm{c}}, \mathcal{B}^{\mathrm{c}}_\dagger, \mathcal{B}^{\mathrm{c},\dagger})$ is hence also Frobenius.*

Proof. The Frobenius property is shown in Proposition 5.9. The statement about the 2-Calabi-Yau property follows directly from Proposition 5.19. □

Remark 5.22. Suppose that $n = 2$. The proof of Proposition 5.9 shows that the suspension functor of the stable ∞-category $\bar{\mathcal{B}}^{\mathrm{c}}$ arising from $(\mathcal{B}^{\mathrm{c}}, \mathcal{B}^{\mathrm{c}}_\dagger, \mathcal{B}^{\mathrm{c},\dagger})$, see Proposition 5.5, acts on objects as the twist functor of the spherical adjunction $G \dashv H$, or by the relative right 2-Calabi-Yau structure equivalently as the delooping (or negative shift) of the Serre functor of $\mathcal{B}$.

It is an interesting problem to determine whether $\bar{\mathcal{B}}^{\mathrm{c}}$ inherits a right 2-Calabi-Yau structure from the relative right 2-Calabi-Yau structure of $\mathcal{B}$.

We conclude this section with the proof of Proposition 5.19.

Proof of Proposition 5.19. Let H be the right adjoint of G and $\mathrm{u}\colon \mathrm{id}_{\mathcal{B}} \to HG$ be the unit. There is a commutative triangle as follows.

$$\begin{tikzcd} \mathrm{Ext}^i_{\mathcal{B}}(\text{-}_1,\text{-}_2) \arrow[r, "G"] \arrow[dr, "\mathrm{u}\circ\text{-}"'] & \mathrm{Ext}^i_{\mathcal{A}}(G(\text{-}_1),G(\text{-}_2)) \arrow[d, "\simeq"] \\ & \mathrm{Ext}^i_{\mathcal{B}}(\text{-}_1,HG(\text{-}_2)) \end{tikzcd} \tag{11}$$

For $X, Y \in \mathcal{B}^c$, the vector space $\mathrm{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y)$ is thus the kernel of the morphism

$$\mathrm{Ext}^i_{\mathcal{B}}(X,Y) \xrightarrow{\mathrm{u}\circ -} \mathrm{Ext}^i_{\mathcal{B}}(X,Y)\,.$$

The relative right n-Calabi–Yau structure of G supplies us with a fiber and cofiber sequence of endofunctors $\mathrm{id}^*_{\mathcal{B}}[-n] \to \mathrm{id}_{\mathcal{B}} \to HG$, see (6). We can evaluate this sequence at any object $Y \in \mathcal{B}^c$ to obtain a fiber and cofiber sequence in $\mathcal{B}^c$. Passing to the corresponding long exact sequence of extensions with $X \in \mathcal{B}^c$ yields the second row in the following diagram in $N(\mathrm{Vect}_k)$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\
\mathrm{Ext}^{i-1}_{\mathcal{B}}(X,HG(Y)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^i_{\mathcal{B}}(X,\mathrm{id}^*_{\mathcal{B}}(Y)[-n]) \arrow[r, "\alpha"] \arrow[d, "\simeq"] & \mathrm{Ext}^i_{\mathcal{B}}(X,Y) \arrow[r, "u\circ(\text{-})"] \arrow[d, "\simeq"] & \mathrm{Ext}^i_{\mathcal{B}}(X,HG(Y)) \arrow[d, "\simeq"] \\
\mathrm{Ext}^{1-i}_{\mathcal{B}}(HG(Y),\mathrm{id}^*_{\mathcal{B}}(X))^* \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Ext}^{-i}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X))^* \arrow[r, "\beta"] \arrow[d] & \mathrm{Ext}^{-i}_{\mathcal{B}}(Y,\mathrm{id}^*_{\mathcal{B}}(X))^* \arrow[r] \arrow[d] & \mathrm{Ext}^{-i}_{\mathcal{B}}(HG(Y),\mathrm{id}^*_{\mathcal{B}}(X))^* \arrow[d] \\
0 \arrow[r] & \mathrm{Ext}^{-i,\mathrm{CY}}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X))^* \arrow[r, "\mathrm{id}"] & \mathrm{Ext}^{-i,\mathrm{CY}}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X))^* \arrow[r] & 0
\end{tikzcd}$$

The top row is obtained by passing to the fiber of $u \circ (\text{-})$, as explained above. The top right square is thus a pullback square.

The third row is obtained from the defining property of the Serre functor $\mathrm{id}^*_{\mathcal{B}}$, see Definition 2.13.

To explain the appearance of the bottom row, we use that G is spherical: The left adjoint F of G is given by $(\mathrm{id}^*_{\mathcal{B}}[1-n])^{-1} \circ H$, where $\mathrm{id}^*_{\mathcal{B}}[1-n]$ is equivalent to the twist functor of the adjunction $G \dashv H$, see [DKSS21, Corollary 2.5.16]. Furthermore by [Chr22b, Lemma 2.11], the morphism (with cofiber $\mathrm{id}_{\mathcal{B}}$)

$$FG \circ \mathrm{id}^*_{\mathcal{B}}(Y)[-n] \simeq HG(Y)[-1] \to \mathrm{id}^*_{\mathcal{B}}(Y)[-n]$$

is a counit morphism of the adjunction $F \dashv G$, and thus adjoint under $FG \dashv HG$ to the unit of $G \dashv H$. It follows that the morphism δ in the above sequence is equivalent to the dual of the morphism

$$\begin{aligned}
\mathrm{Ext}^{-i}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X)) \xrightarrow{\mathrm{u}\circ -} & \mathrm{Ext}^{-i}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n], HG \circ \mathrm{id}^*_{\mathcal{B}}(X))\,. \\
\simeq\ & \mathrm{Ext}^{-i}_{\mathcal{B}}(FG \circ \mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X)) \\
\simeq\ & \mathrm{Ext}^{-i}_{\mathcal{B}}(HG(Y)[-1],\mathrm{id}^*_{\mathcal{B}}(X)) \\
\simeq\ & \mathrm{Ext}^{1-i}_{\mathcal{B}}(HG(Y),\mathrm{id}^*_{\mathcal{B}}(X))
\end{aligned}$$

arising from postcomposition with the unit u of $G \dashv H$. By the commutativity of the diagram (11), this shows that $\mathrm{Ext}^{-i}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X))$ fits into the above diagram as the indicated pushout.

Finally we note that, since any exact sequence in Vect_k splits, $\mathrm{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y)$ is the maximal direct summand on which α restricts to an isomorphism and $\mathrm{Ext}^{-i,\mathrm{CY}}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X))^*$ is the maximal direct summand on which β restricts to an isomorphism. This shows that the arising vertical morphism

$$\mathrm{Ext}^{i,\mathrm{CY}}_{\mathcal{B}}(X,Y) \to \mathrm{Ext}^i_{\mathcal{B}}(X,Y) \simeq \mathrm{Ext}^{-i}_{\mathcal{B}}(Y,\mathrm{id}^*_{\mathcal{B}}(X))^* \to \mathrm{Ext}^{-i,\mathrm{CY}}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X))^*$$

is an isomorphism. To conclude, we note that

$$\mathrm{Ext}^{-i,\mathrm{CY}}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y)[-n],\mathrm{id}^*_{\mathcal{B}}(X))^* \simeq \mathrm{Ext}^{n-i,\mathrm{CY}}_{\mathcal{B}}(\mathrm{id}^*_{\mathcal{B}}(Y),\mathrm{id}^*_{\mathcal{B}}(X))^*\,.$$

□

6 The geometric model

In this section, we describe a geometric model for the cosingularity category $\mathcal{C}_{\mathbf{S}}$, including a classification of all indecomposable objects in terms so called matching curves, which we introduce in Section 6.1. In Section 6.2, we recall the results from [Chr21], which associate objects of $\mathcal{C}_{\mathbf{S}}$ to matching curves and describe the Homs in terms of the intersections of the curves. In Section 6.3, we compare the computation of cones in $\mathcal{C}_{\mathbf{S}}$ of morphisms arising from intersections with the Kauffman skein relation. The more technical aspects are discussed in Sections 6.4 and 6.5. In Section 6.4, we give a review of the construction of the objects from matching curves in [Chr21]. In the final Section 6.5, we prove the geometrization Theorem 6.13, which states that every compact object in $\mathcal{C}_{\mathbf{S}}$ decomposes uniquely into the direct sum of objects associated to matching curves.

For the remainder of this article, we will always assume that k is an algebraically closed field. For the entirety of this section, we also fix a marked surface $\mathbf{S}$ and an auxiliary trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$.

6.1 Matching curves

Definition 6.1. An allowed curve is a continuous map $\gamma\colon U \to \mathbf{S}\backslash M$ with $U = [0,1], S^1$, satisfying that

1. all existent endpoints of γ (possibly none) lie in $\partial\mathbf{S}\backslash M$.
2. away from the endpoints, γ is disjoint from $\partial\mathbf{S}$.
3. γ does not cut out an unmarked disc[2] in $\mathbf{S}$.
4. if $U = S^1$, then γ is not homotopic to the composite of multiple identical closed curves.

An open matching curve γ in $\mathbf{S}$ is an equivalence class of allowed curves under homotopies relative $\partial\mathbf{S}\backslash M$ with domain $U = [0,1]$ and considered up to reversal of orientation. We say that the rank of γ is 1.

A closed matching curve γ in $\mathbf{S}$ is an equivalence class of allowed curves under homotopies relative $\partial\mathbf{S}\backslash M$ with domain $U = S^1$ together with choices of an integer $a \geq 1$ and $\lambda \in k^\times$. We call a the rank of γ and λ the monodromy datum[3]. We consider closed matching curves up to simultaneously reversing the orientation and replacing the monodromy datum λ by its inverse λ^{-1}. We say that two closed matching curves are distinct if either their underlying curves or their monodromy datum differ.

Definition 6.2. Let $\gamma : U \to \mathbf{S}$, $\gamma'\colon U' \to \mathbf{S}$ be two matching curves. We choose representatives of γ and γ' with the minimal number of intersections.

- A crossing of γ and γ' is an intersection of γ and γ' away from their endpoints. We denote the set of crossings of γ and γ' by $i^{\mathrm{cr}}(\gamma,\gamma')$. If $\gamma = \gamma'$, then $i^{\mathrm{cr}}(\gamma,\gamma)$ counts each self-crossing only once.
- A directed boundary intersection from γ to γ' is an intersection of both γ and γ' with the same connected component B of $\partial\mathbf{S}\backslash M$ such that the intersection of γ' with B follows the intersection of γ with B in the orientation of B induced by the (clockwise) orientation of $\mathbf{S}$. We denote the set of directed boundary intersection from γ to γ' by $i^{\mathrm{bdry}}(\gamma,\gamma')$. If $\gamma = \gamma'$, then $i^{\mathrm{bdry}}(\gamma,\gamma)$ only counts those directed boundary intersections arising from two distinct ends of γ lying on the same component of $\partial\mathbf{S}\backslash M$.

Definition 6.3. An open matching curve $\gamma\colon [0,1] \to \mathbf{S}$ is called an arc if it has no self-crossings. An arc is called a boundary arc, if it cuts out a monogon. An arc is called an internal arc, if it is not a boundary arc.

Remark 6.4. The notion of an internal arc in the sense of Definition 6.3 coincides with the notion of an arc from [FT18, Def. 5.2], except that we use a different convention regarding

[2]By cutting out, we mean that a connected part of the curve bounds an unmarked disc.
[3]The corresponding monodromy matrix is the $a \times a$-Jordan block with eigenvalue λ.

Figure 1: A crossing and a direct endpoint intersection of two matching curves. The boundary of the surface is depicted in green. The magenta arrow indicates the clockwise direction. The orange vertex describes a marked point.

endpoints. In this paper, endpoints lie in $\partial\mathbf{S}\backslash M$, whereas in loc. cit. endpoints lie in M. These two perspectives are however equivalent, as can be seen by expanding the marked points to intervals, contracting their complements to points and using that arcs are considered up to homotopy.

Definition 6.5 (See e.g. [FST08])**.** Two arcs in $\mathbf{S}$ are called compatible, if they do not have any crossings. An ideal triangulation of $\mathbf{S}$ consists of a maximal (by inclusion) collection I of pairwise compatible arcs in $\mathbf{S}$.

We note that any ideal triangulation of $\mathbf{S}$ contains all boundary arcs of $\mathbf{S}$ and that any two ideal triangulations of $\mathbf{S}$ have the same cardinality.

6.2 Objects and Homs

The cosingularity category $\mathcal{C}_{\mathbf{S}}$ embeds fully faithfully into the ∞-category $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})$ of global sections of the perverse schober $\mathcal{F}_{\mathcal{T}}$ as the full subcategory spanned by global sections whose value at any edge $e\in\mathcal{T}_1$ lies in $\mathcal{D}(k[t_1^{\pm}])\subset\mathcal{D}(k[t_1])=\mathcal{F}_{\mathcal{T}}(e)$. To describe the objects and morphism objects in the cosingularity category $\mathcal{C}_{\mathbf{S}}$, we may thus make use of the partial geometric model for $\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})$ of [Chr21].

Consider a matching curve γ in $\mathbf{S}$. Such a curve defines by [Chr21, Lemma 4.8] a matching datum $(\gamma,k[t_1^{\pm}])$ in the sense of [Chr21, Def. 4.9] with γ a finite, pure matching curve in $\mathbf{S}\backslash M$ in the sense of [Chr21, Def. 4.5]. By the results of Section 4.2. in loc. cit., we thus get for each matching curve γ in $\mathbf{S}$ a global section $M_\gamma\coloneqq M_\gamma^{k[t_1^{\pm}]}\in\mathcal{C}_{\mathbf{S}}\subset\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})$. We further have the following.

Proposition 6.6 ([Chr21, Cor. 5.14])**.** *Let γ be a matching curve in $\mathbf{S}$. Then the discrete endomorphism ring of $M_\gamma\in\mathcal{C}_{\mathbf{S}}$ is local and M_γ thus indecomposable.*

We furthermore note that the assignment of global sections with matching curves is independent of the chosen 3-valent spanning graph $\mathcal{T}$:

Remark 6.7. The assignment of isomorphism classes of global sections with matching curves does not depend on the choice of chosen trivalent spanning graph $\mathcal{T}$. This invariance can be seen as follows:

Let $\mathcal{T},\mathcal{T}'$ be two 3-valent spanning graphs of $\mathbf{S}$. Then the corresponding dual ideal triangulations differ by a finite sequence of flips. Associated with the sequence of flips is an equivalence of ∞-categories $E\colon\mathcal{H}(\mathcal{T},\mathcal{F}_{\mathcal{T}})\simeq\mathcal{H}(\mathcal{T}',\mathcal{F}_{\mathcal{T}'})$, see [Chr21, Section 4.4]. For matching data $(\gamma,k[t_1^{\pm}])$ with local value $L=k[t_1^{\pm}]$, the resulting global section does not change under homotopies of γ that are allowed to cross the trivalent vertices of $\mathcal{T}$, see [Chr21, Rem. 4.17]. Theorem 4.21 in [Chr21] states that for a matching datum $(\gamma,k[t_1^{\pm}])$, the image of $M_\gamma^{k[t_1^{\pm}]}$ under E is indeed given by $M_{\gamma'}^{k[t_1^{\pm}]}$ with γ' homotopic to γ in the above sense.

The morphism objects between the M_γ's can be described in terms of intersections of the curves as in the following two Theorems, see [Chr21, Theorems 5.4 and 5.5][4].

[4]We note that in Theorem 6.8 the case that the curves underlying γ,γ' agree but their monodromy data differ

Theorem 6.8. *Let γ, γ' be two distinct matching curves in* $\mathbf{S}$. *Let a be the rank of γ and a' the rank of γ'. There exists an equivalence in $\mathcal{D}(k[t_2^{\pm}])$*

$$\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) \simeq k[t_1^{\pm}]^{\oplus aa' i^{\mathrm{cr}}(\gamma,\gamma') \oplus i^{\mathrm{bdry}}(\gamma,\gamma')}\,.$$

Theorem 6.9. *(i) Let $\gamma\colon [0,1] \to \mathbf{S}$ be an open matching curve in* $\mathbf{S}$. *There exists an equivalence in $\mathcal{D}(k[t_2^{\pm}])$*

$$\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma) \simeq k[t_1^{\pm}]^{\oplus 1+2i^{\mathrm{cr}}(\gamma,\gamma)+i^{\mathrm{bdry}}(\gamma,\gamma)}\,.$$

(ii) Let $\gamma\colon S^1 \to \mathbf{S}$ be a closed matching curve in $\mathbf{S}$ *of rank a. There exists an equivalence in $\mathcal{D}(k[t_2^{\pm}])$*

$$\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma) \simeq k[t_1^{\pm}]^{\oplus 2a+2a^2 i^{\mathrm{cr}}(\gamma,\gamma)}\,.$$

6.3 Skein relations from mapping cones

In the following, we collect geometric descriptions of the cones of the morphisms between the objects of $\mathcal{C}_{\mathbf{S}}$ associated to the matching curves of rank 1. Similar descriptions of cones in the derived categories of gentle algebras appear in [OPS18]. We fix two matching curves γ, γ' in $\mathbf{S}$ and distinguish two cases.

Case 1: γ and γ' have a crossing.

There are two possible *smoothings* of this crossing. To describe the smoothings, we first suppose that both γ and γ' are open. In this case, each of the two smoothings consisting of two matching curves in $\mathbf{S}$, denoted γ_1, γ_2 and γ_3, γ_4. The first curve in each of the two smoothings is obtained by starting at an endpoint of γ and tracing along γ up to that crossing, and then tracing along γ' in one of two possible directions. Similarly, the second curve in the two smoothings is obtained by starting at the other endpoint of γ, tracing along γ up to the crossing, and then tracing along γ' to the other end. This process is locally at the crossing illustrated on the left in Figure 2.

If one of the two matching curves γ, γ' is closed, say γ', the smoothings are instead given by single open matching curves, which we denote by γ_1, γ_2. These are obtained by starting at an endpoint of γ, tracing along γ up to the intersection point, then tracing once around γ' in one of two chosen directions, and finally tracing along γ from the intersection point to its other endpoint. The local picture of the smoothing at the crossing in Figure 2 still applies in this case, but γ_1, γ_2 in the figure join into one curve γ_1, and γ_3, γ_4 join into one curve γ_2.

Finally, suppose that γ, γ' are closed, of rank 1. The two possible smoothings are given by two rank 1 closed matching curves γ_1, γ_2, which are obtained as before by applying the local smoothing operation depicted in Figure 2. Their monodromy data are defined as follows. We choose orientations for γ, γ' which specifies their monodromy data, denoted λ, λ'. We can obtain either smoothing by partially tracing along γ along its orientation. In the smoothing of the crossing that traces along γ' along its orientation, the monodromy datum is defined as $\lambda \cdot \lambda'$. For the other smoothing, the monodromy datum is defined as $\lambda \cdot (\lambda')^{-1}$. One readily checks that the assignment is well defined, under compatible reversals of the orientations and changes in monodromy data of the involved curves according to Definition 6.1.

Case 2: there is a directed boundary intersection from γ to γ'. In this case γ and γ' must be open.

We can compose γ with part of a boundary component of $\mathbf{S}\backslash M$ and γ' to a curve, which we then smooth to a matching curve γ_1. This process is illustrated on the right in Figure 2.

To both types of intersection, Theorems 6.8 and 6.9 associate a direct summand of the morphism object $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$ and also $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$ in case of a crossing. Proposition 6.10 describes the cones of these morphisms in terms of the above smoothings.

is not stated in [Chr21]. Inspecting the proof of [Chr21, Thm. 5.5], see specifically Equation (38), one arrives at the given description in this case.

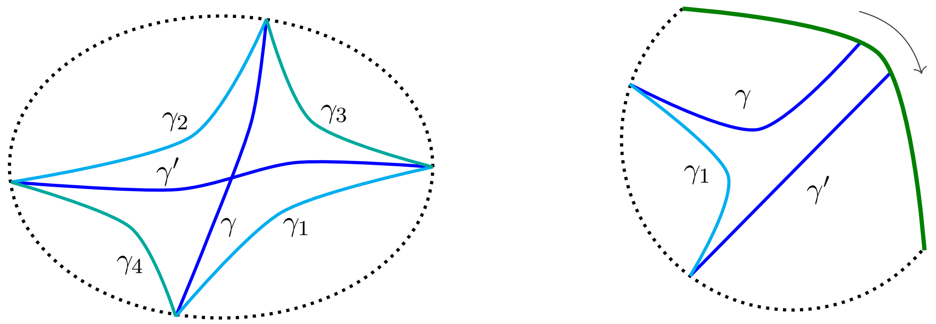


Figure 2: On the left: a crossing of two matching curves γ, γ (in blue) and the two possible smoothings γ_1, γ_2 and γ_3, γ_4. On the right: a directed boundary intersection of two matching curves γ, γ' and the corresponding smoothed composite γ_1. The boundary of $\mathbf{S}$ is depicted in green. Outside of the depicted parts of $\mathbf{S}$, the matching curves continue identically.

Proposition 6.10. *Let γ, γ' be two matching curves in $\mathbf{S}$ of rank 1.*

(1) Suppose that γ and γ' are open and have a crossing. There exist fiber and cofiber sequences in $\mathcal{C}_{\mathbf{S}}$

$$M_{\gamma_1} \oplus M_{\gamma_2} \to M_\gamma \xrightarrow{\alpha} M_{\gamma'}\,, \qquad M_{\gamma_3} \oplus M_{\gamma_4} \to M_{\gamma'} \xrightarrow{\beta} M_\gamma\,,$$

with γ_1, γ_2 and γ_3, γ_4 being the two possible smoothings of the crossing. The morphisms α and β describe any non-zero degree 0 elements of the direct summands $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$, $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_{\gamma'}, M_\gamma)$ associated to the crossing in Theorem 6.8.

(2) Suppose that γ' is closed and that γ, γ' have a crossing. There exist fiber and cofiber sequences in $\mathcal{C}_{\mathbf{S}}$

$$M_{\gamma_1} \to M_\gamma \xrightarrow{\alpha} M_{\gamma'}\,, \qquad M_{\gamma_2} \to M_{\gamma'} \xrightarrow{\beta} M_\gamma\,,$$

with γ_1 and γ_2 being the two possible smoothings of the crossing. The morphisms α and β describe any non-zero degree 0 elements of the direct summands $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$, $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_{\gamma'}, M_\gamma)$ associated to the crossing in Theorem 6.8.

(3) Suppose that there is a directed boundary intersection from γ to γ'. There exist a fiber and cofiber sequence in $\mathcal{C}_{\mathbf{S}}$

$$M_{\gamma_1} \to M_\gamma \xrightarrow{\alpha} M_{\gamma'}\,,$$

with γ_1 the smoothed composite of γ, γ'. The morphism α describes any non-zero degree 0 element of the direct summand $k[t_1^\pm] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$ associated to the directed boundary intersection in Theorem 6.8.

Proof. We will prove the assertion via a direct computation, using the descriptions of M_γ and $M_{\gamma'}$ as coCartesian sections of the Grothendieck construction of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$. We first describe the computation for part (1) and then describe how to adapt it for part (2). The computations for part (3) are similar and left to the reader.

We first show part (1) in the case that γ, γ' are embedded arcs with a single crossing. By Remark 6.7, the assertion does not depend on the choice of the trivalent spanning graph $\mathcal{T}$. We can choose $\mathcal{T}$ such that the crossings happens in a 4-gon in the corresponding ideal triangulation where the curves γ, γ' enter and exit through all four boundary edges. This is achieved by considering the restrictions of the four curves $\gamma_1, \dots, \gamma_4$ to the neighborhood of the crossing, then extending them to four compatible arcs forming a 4-gon and finally choosing an ideal triangulation containing these four arcs.

The local model for perverse schobers in Section 3.2 implies that $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ is locally on the 4-gon equivalent to the following diagram

$$\begin{tikzcd}[column sep=small]
\mathcal{D}(k[t_1^{\pm}]) & & & & \mathcal{D}(k[t_1^{\pm}]) \\
& \mathrm{Fun}(\Delta^1,\mathcal{D}(k[t_1^{\pm}])) \arrow[ul, "\mathrm{cof}"'] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & \mathrm{Fun}(\Delta^1,\mathcal{D}(k[t_1^{\pm}])) \arrow[ur, "\mathrm{cof}"] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & \\
\mathcal{D}(k[t_1^{\pm}]) & & \mathcal{D}(k[t_1^{\pm}]) & & \mathcal{D}(k[t_1^{\pm}])
\end{tikzcd}$$

which describes a perverse schober parametrized by the ribbon graph with two 3-valent vertices connected by an edge.

Inspecting their construction in [Chr21], we see that the global sections M_γ and $M_{\gamma'}$ are locally at the 4-gon given by the following coCartesian sections:

$$M_\gamma = \begin{tikzcd}[column sep=small]
0 & & & & k[t_1^{\pm}][1] \\
& k[t_1^{\pm}] \xrightarrow{\mathrm{id}} k[t_1^{\pm}] \arrow[ul, "\mathrm{cof}"'] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & k[t_1^{\pm}] \to 0 \arrow[ur, "\mathrm{cof}"] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & \\
k[t_1^{\pm}] & & k[t_1^{\pm}] & & 0
\end{tikzcd}$$

$$M_{\gamma'} = \begin{tikzcd}[column sep=small]
k[t_1^{\pm}] & & & & 0 \\
& 0 \to k[t_1^{\pm}] \arrow[ul, "\mathrm{cof}"'] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & k[t_1^{\pm}] \xrightarrow{\mathrm{id}} k[t_1^{\pm}] \arrow[ur, "\mathrm{cof}"] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & \\
0 & & k[t_1^{\pm}] & & k[t_1^{\pm}]
\end{tikzcd}$$

Note that $k[t_1^{\pm}][1] \simeq k[t_1^{\pm}]$.

Limits in the ∞-category of coCartesian sections are computed pointwise in $\mathrm{Exit}(\mathcal{T})$, as follows from [Lur09, 5.1.2.3, 4.3.1.10]. The morphism $M_{\gamma'} \to M_\gamma$ evaluates for instance at the left vertex to the 'inclusion' $(0 \to k[t_1^{\pm}]) \to (k[t_1^{\pm}] \xrightarrow{\mathrm{id}} k[t_1^{\pm}])$ in $\mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}]))$ and at the right vertex to the 'projection' $(k[t_1^{\pm}] \xrightarrow{\mathrm{id}} k[t_1^{\pm}]) \to (k[t_1^{\pm}] \to 0)$, with respective fibers $(k[t_1^{\pm}][-1] \to 0) \simeq (k[t_1^{\pm}] \to 0)$ and $(0 \to k[t_1^{\pm}])$. In total, the fiber of the morphism $M_{\gamma'} \to M_\gamma$ is locally on the 4-gon given by the coCartesian section

$$\mathrm{fib}(M_{\gamma'} \to M_\gamma) = \begin{tikzcd}[column sep=small]
k[t_1^{\pm}] & & & & k[t_1^{\pm}] \\
& k[t_1^{\pm}] \to 0 \arrow[ul, "\mathrm{cof}"'] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & 0 \to k[t_1^{\pm}] \arrow[ur, "\mathrm{cof}"] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & \\
k[t_1^{\pm}] & & 0 & & k[t_1^{\pm}]
\end{tikzcd}$$

and is away from the 4-gon identical to $M_{\gamma'} \oplus M_\gamma[-1]$. We thus see that $\mathrm{fib}(M_{\gamma'} \to M_\gamma) \simeq M_{\gamma_3} \oplus M_{\gamma_4}$ with γ_3, γ_4 as depicted in Figure 2.

The morphism $M_\gamma \to M_{\gamma'}$ evaluates trivially at all objects of $\mathrm{Exit}(\mathcal{T})$ but is nevertheless non-trivial, with the non-triviality encoded in two null-homotopies of the evaluation of the morphism at middle edge, see case 2) in the section on crossings in [Chr21, Section 5.2]. The fiber is the coCartesian section that is locally given as

$$\begin{tikzcd}[column sep=small]
k[t_1^{\pm}] & & & & k[t_1^{\pm}] \\
& k[t_1^{\pm}] \xrightarrow{(\mathrm{id},0)} k[t_1^{\pm}] \oplus k[t_1^{\pm}] \arrow[ul, "\mathrm{cof}"'] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & k[t_1^{\pm}] \oplus k[t_1^{\pm}] \xrightarrow{(\mathrm{id},0)} k[t_1^{\pm}] \arrow[ur, "\mathrm{cof}"] \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & \\
k[t_1^{\pm}] & & k[t_1^{\pm}] \oplus k[t_1^{\pm}] & & k[t_1^{\pm}]
\end{tikzcd}$$

which again matches $M_{\gamma_1} \oplus M_{\gamma_2}$. To illustrate why the above describes the fiber, we can use that the ∞-category of coCartesian sections as depicted is equivalent to

$$\mathrm{Fun}(\Delta^2, \mathcal{D}(k[t_1^\pm]) \simeq \mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^\pm]) \times_{\mathcal{D}(k[t_1^\pm]} \mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^\pm])\,.$$

Under this equivalence, the fiber and cofiber sequence $M_{\gamma'}[-1] \to \mathrm{fib}(M_\gamma \to M_{\gamma'}) \to M_\gamma$, where $M_{\gamma'}[-1] \simeq M_{\gamma'}$, amounts to the apparent fiber and cofiber sequence

$$(0 \to k[t_1^\pm] \xrightarrow{\mathrm{id}} k[t_1^\pm]) \to (k[t_1^\pm] \xrightarrow{\mathrm{id}} k[t_1^\pm] \xrightarrow{\mathrm{id}} k[t_1^\pm]) \oplus (0 \to k[t_1^\pm] \to 0) \to (k[t_1^\pm] \xrightarrow{\mathrm{id}} k[t_1^\pm] \to 0)\,.$$

The above computation generalizes as follows to the case that γ, γ' are not arcs and are allowed multiple crossings. Before and after the crossing, γ, γ' are composed of a collection of common segments (in the above case this collection is empty). The segments before and after the crossing then differ and are arranged as in the above case. See also the discussion of crossings in [Chr21, Section 5.2]. One of the two crossing morphisms restricts to the identity on the parts of the global sections arising from the segments, whereas the other morphism vanishes. The fiber is then computed as above, on the part of the global sections arising from the involved segments, with the difference that the diagrams involve equivalent middle parts arising from the common segments at the crossing.

Since the computation for part (1) is local in nature, it also applies to part (2). The only thing left to verify is that the given monodromy data of the smoothings correctly describe the fibers. This follows from unraveling the description of monodromy data given in case 2) of part 1) of the proof of Theorem 6.13 below: the monodromy datum of the smoothing is obtained from tracing a choice of basis of the stalk value along the functors defining $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$, as well as their adjoints. Locally at the crossing, these identifications can be chosen trivially (as seen in the above local model). The changes in basis of the two curves γ, γ' is thus composed when tracing along the smoothings. Since by assumption the ranks are 1, the basis has a single element and we simply multiply the monodromy data. □

Remark 6.11. Parts (1) and (2) of Proposition 6.10 match the $q = 1$ Kauffman skein relations, or for arcs equivalently the Ptolemy cluster exchange relations, see Definitions 7.7 and 7.10.

6.4 Review of the gluing construction of global sections

Before we state and prove the geometrization Theorem in Section 6.5, we recall the construction of the global sections from matching curves in [Chr21].

Definition 6.12. Let v be a vertex of $\mathcal{T}$. A segment at v is an embedded curve $\delta\colon [0,1] \to \mathbf{S}$ lying in a contractible neighborhood of v whose endpoints lie on two distinct edges of $\mathcal{T}$ incident to v (but not on v).

We consider segments at a vertex v of $\mathcal{T}$ as equivalence classes under homotopies $\Delta\colon [0,1]^2 \to \mathbf{S}$, which satisfy that $\Delta(t)$ is a segment at v for all $t \in [0,1]$. A segment in $\mathbf{S}$ is a segment at any vertex v of $\mathcal{T}$.

It is straightforward to see that every matching curve in $\mathbf{S}$ arises in a unique way by composing finitely many segments in $\mathbf{S}$, such that whenever two segments are composed, they lie at distinct vertices of $\mathcal{T}$.

To each segment δ at a vertex v of $\mathcal{T}$, there is an associated lax section $M_\delta \in \mathcal{L}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}})$ in the sense of Definition 3.9. Such a lax section M can be identified with the data of

- for each object $x \in \mathrm{Exit}(\mathcal{T})$ an object $M(x)$ in the stable ∞-category $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(x)$ and
- for each morphism $x \to y$ in $\mathrm{Exit}(\mathcal{T})$ a morphism $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(x \to y)(M(x)) \to M(y)$ in $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(y)$. If all these morphisms are equivalences (hence describe coCartesian morphisms), the lax section is a global section.

The lax section M_δ associated to the segment δ is a left Kan extension relative the Grothendieck construction $p\colon \Gamma(\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}) \to \mathrm{Exit}(\mathcal{T})$ along the inclusion $\{v\} \hookrightarrow \mathrm{Exit}(\mathcal{T})$ of $M_\delta(v) \in \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(v)$. In particular, this lax section is fully determined by its value $M_\delta(v)$ at $v \in \mathrm{Exit}(\mathcal{T})$. Spelling

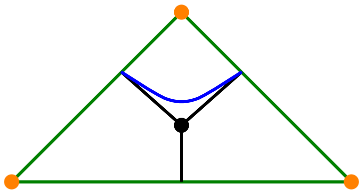


Figure 3: One of the three possible segments (in blue) at a vertex of $\mathcal{T}$.

out the Kan extension, one finds that for each edge e incident to v, $M_\delta(e)$ is equivalent to the image of $M_\delta(v)$ under the functor $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}(v \to e)$. Further, if $x \in \mathrm{Exit}(\mathcal{T})$ is neither v not an edge incident to v, then $M_\delta(x)$ vanishes.

Let v be a vertex of $\mathcal{T}$. There are three indecomposable objects in $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}(v) \simeq \mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}]))$, given as follows:

$$\left(0 \to k[t_1^{\pm}]\right)\ ,\ \left(k[t_1^{\pm}] \xrightarrow{\simeq} k[t_1^{\pm}]\right)\ ,\ \left(k[t_1^{\pm}] \to 0\right)\ .$$

These three objects describe the values of the three lax sections associated to the three possible segments at v. Let e_1, e_2, e_3 be the three edges incident to v. Each of the above objects satisfies that applying $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}(v \to e_i)$ yields $k[t_1^{\pm}]$ in two cases and 0 in one case. We choose the value $M_\delta(v)$ so that the set of edges of $\mathcal{T}$ where M_δ evaluates nontrivially as $k[t_1^{\pm}]$ consists exactly of the two edges at which δ begins and ends.

To produce a global section associated to a matching curve γ, we glue the lax sections associated to the segments of γ. In the simplest case of gluing sections from two segments, this goes as follows. Let e be an edge of $\mathcal{T}$ with incident vertices v, v' and let δ and δ' be two segments at v, respectively v', which end at e. We have $M_\delta(e) \simeq M_{\delta'}(e) \simeq k[t_1^{\pm}]$. Define $Z_e \in \mathcal{L}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}})$ via $Z_e(x) = 0$ if $x \neq e$ and $Z_e(e) = k[t_1^{\pm}]$. There are apparent pointwise inclusions of Z_e into M_δ and $M_{\delta'}$, which restrict at e to the identity on $k[t_1^{\pm}]$. The gluing of M_δ and $M_{\delta'}$ is now defined as the pushout in $\mathcal{L}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}})$ of the following diagram.

$$\begin{tikzcd} & Z_e \arrow[dl] \arrow[dr] & \\ M_\delta & & M_{\delta'} \end{tikzcd} \tag{12}$$

More generally, given any curve η composed of segments, we can glue a section $M_\eta \in \mathcal{L}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}})$ by repeatedly taking pushouts as above. If η is an open matching curve, the section M_η is even a global sections, i.e. lies in $\mathcal{H}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}) \subset \mathcal{L}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}})$.

Consider finally a closed matching curve γ of rank a with monodromy datum $\lambda \in k^\times$. Let $\mathscr{J}_\lambda \colon k[t_1^{\pm}]^{\oplus a} \simeq k[t_1^{\pm}]^{\oplus a}$ be the $a \times a$-Jordan block with eigenvalue λ. We cut γ at any edge of $\mathcal{T}$ into an open curve η which is composed of segments. The global section $M_\gamma \in \mathcal{H}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}) \subset \mathcal{L}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}})$ is obtained as the coequalizer of

$$\begin{tikzcd} Z_e^{\oplus a} \arrow[r, shift left, "\iota_1^{\oplus a}"] \arrow[r, shift right, "\iota_2^{\oplus a} \circ \mathscr{J}_\lambda^{-1}"'] & M_\eta^{\oplus a} \end{tikzcd} \tag{13}$$

in $\mathcal{L}(\mathcal{T}, \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}})$. Here, ι_1, ι_2 are two pointwise inclusions of Z_e into M_η at e arising from the two ends of η at e. In the way we state this definition, making different choices of Kan extensions in the construction of M_η changes the section M_γ. One can show that the difference between the M_γ's associated to two possible choices is the same as multiplying the monodromy datum λ of γ by a non-zero factor. One can solve this by fixing choices of these Kan extensions. Alternatively, the proof of the geometrization Theorem 6.13 also shows how to extract a unique monodromy datum from a global section of the form M_γ.

We note that we have defined matching curves as certain curves considered up to reversal of orientation. If γ is an open matching curve, it is easy to see that the global section M_γ

does not depend on the orientation of γ. If γ is a closed matching curve, changing the orientation of γ in the construction of M_γ interchanges the two maps in (13). Hence, to obtain the same global section, we need to replace the monodromy datum by its inverse and use that the Jordan normal form of $\mathscr{J}_\lambda^{-1}$ is a single Jordan block with eigenvalue λ^{-1} .

6.5 The geometrization Theorem

The main result of this section is to prove the following Theorem, stating that all compact objects in $\mathcal{C}_{\mathbf{S}}$ are geometric, meaning that they arise as direct sums of object associated with matching curves.

Theorem 6.13 (The geometrization Theorem)**.** *Let $X \in \mathcal{C}_{\mathbf{S}}$ be a compact object. Then there exists a unique finite family $\{\gamma_j\}_{j\in J}$ of matching curves in $\mathbf{S}$ satisfying that there is an equivalence in $\mathcal{C}_{\mathbf{S}}$*

$$X \simeq \bigoplus_{j\in J} M_{\gamma_j} \,.$$

Corollary 6.14. *The full subcategory $\mathcal{C}_{\mathbf{S}}^{c} \subset \mathcal{C}_{\mathbf{S}}$ of compact objects is Krull-Schmidt, meaning that every objects splits into a direct sum of objects with local endomorphism rings.*

Proof of Corollary 6.14. As shown in the proof of Theorem 6.13, every compact object X of $\mathcal{C}_{\mathbf{S}}$ splits as the direct sum $X \simeq \bigoplus_{\gamma\in J} M_\gamma$, with J a set of matching curves. The endomorphism rings of these objects are local by Proposition 6.6. □

The proof of the geometrization Theorem makes use of the gluing description of global sections arising from matching curves and the description of global sections in terms of coCartesian sections of the Grothendieck construction. Given a compact object X of $\mathcal{C}_{\mathbf{S}}$, considered as a global section of $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$, we can evaluate it at any edge e of the ribbon graph $\mathcal{T}$. If the value is non-zero we can choose a direct sum decomposition of this value with a non-zero direct summand. Let v be a vertex incident to e. As we show in Construction 6.15, we can almost find a lift along the functor $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(v \to e)$ of the direct sum decomposition at e to a direct sum decomposition of the value of X to v. This then give rise to direct sum decompositions at further edges incident to v. Proceeding this way, we choose repeated direct sum decompositions of the values of X at all vertices and edges of $\mathcal{T}$. We then observe that we can further tweak the chosen direct sum decompositions so that they are all compatible and thus glue to a global section, which is a non-zero direct summand of X. The summand of X so constructed is obtained from gluing lax sections associated to segments and hence geometric, i.e. arises from a matching curve. Repeating this argument, we find a splitting of X into geometric direct summands. The uniqueness of the family J in the geometrization Theorem is proven separately.

Construction 6.15. Let $\mathbf{S}$ be the 3-gon and e an edge of a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$. Let $X \in \mathcal{C}_{\mathbf{S}}^{c}$ and let $\mathrm{ev}_e(X) \simeq B \oplus A \oplus B'$ with $A \neq 0$. We construct two direct sum decompositions $X \simeq Z \oplus Y \oplus Z'$ and $X \simeq \tilde{Z} \oplus Y \oplus \tilde{Z}'$, called the counterclockwise and clockwise splittings. These satisfy that the arising equivalences

$$\mathrm{ev}_e(Z) \oplus \mathrm{ev}_e(Y) \oplus \mathrm{ev}_e(Z') \simeq B \oplus A \oplus B'$$

$$\mathrm{ev}_e(\tilde{Z}) \oplus \mathrm{ev}_e(Y) \oplus \mathrm{ev}_e(\tilde{Z}') \simeq B \oplus A \oplus B'$$

are lower triangular with invertible diagonals.

The cosingularity category $\mathcal{C}_{\mathbf{S}}$ of $\mathbf{S}$ is equivalent to the functor category $\mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}]))$, whose objects are diagrams $x \to y$ with $x, y \in \mathcal{D}(k[t_1^{\pm}])$. In the following, we set $\mathcal{C}_{\mathbf{S}} = \mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}]))$. The functor $\mathrm{ev}_e \colon \mathcal{C}_{\mathbf{S}} \to \mathcal{D}(k[t_1^{\pm}])$ can be chosen to evaluate a functor $\Delta^1 \to \mathcal{D}(k[t_1^{\pm}])$ at its value at $1 \in \Delta^1$ (recall that Δ^1 has the vertices $0, 1$). There are three indecomposable objects in $\mathcal{C}_{\mathbf{S}}$, corresponding to the three boundary arcs in $\mathbf{S}$ (there are no further matching curves in $\mathbf{S}$). Explicitly, these objects can be described as follows.

$$\begin{aligned} M_1 &= \left(0 \to k[t_1^{\pm}]\right) \\ M_2 &= \left(k[t_1^{\pm}] \xrightarrow{\simeq} k[t_1^{\pm}]\right) \\ M_3 &= \left(k[t_1^{\pm}] \to 0\right) \end{aligned}$$

The homotopy category of $\mathcal{D}(k[t_1^{\pm}])$ is equivalent to the abelian 1-category of k-vector spaces, see Lemma 2.11. We thus treat objects in $\mathcal{D}(k[t_1^{\pm}])$ as vector spaces in the following.

All compact objects in $\mathcal{D}(k[t_1^{\pm}])$ are a finite direct sums of copies of $k[t_1^{\pm}]$. Using that every morphism in $\mathcal{D}(k[t_1^{\pm}])$ splits, we may assume that the object $X \in \mathcal{C}_{\mathbf{S}}$ is equivalent to an object of the form

$$k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus l} \xrightarrow{\mathscr{M}} k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus i}\,,$$

with $\mathscr{M} = \begin{pmatrix} \mathrm{id}_{k[t_1^{\pm}]^{\oplus j}} & 0 \\ 0 & 0 \end{pmatrix}$ and $i, j, l \geq 0$. We thus have $\mathrm{ev}_e(X) = k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus i}$ and a splitting

$$X \simeq \left(k[t_1^{\pm}]^{\oplus l} \to 0\right) \oplus \left(k[t_1^{\pm}]^{\oplus j} \hookrightarrow k[t_1^{\pm}]^{\oplus j} \oplus k[t_1^{\pm}]^{\oplus i}\right)\,.$$

We denote the first summand by O and the second by U. Note that $\mathrm{ev}_e(O) = 0$.

Let $C = (B \oplus A) \cap k[t_1^{\pm}]^{\oplus j}$. We define V as the diagram $C \to B \oplus A$ in $\mathcal{D}(k[t_1^{\pm}])$. It is easy to see that the apparent morphism $V \to U$ in $\mathcal{C}_{\mathbf{S}}$ admits a retraction. We can thus choose a splitting $U \simeq V \oplus Q'$. By construction, we have $\mathrm{ev}_e(V) = B \oplus A$. Next, we define $D = B \cap k[t_1^{\pm}]^{\oplus j}$. Then we again obtain a direct summand $Q = (D \to B) \hookrightarrow V$. We choose a direct sum complement $V \simeq Q \oplus Y$. We have $\mathrm{ev}_e(Q) \oplus \mathrm{ev}_e(Y) \oplus \mathrm{ev}_e(Q') \simeq B \oplus A \oplus B'$ and this equivalence is lower triangular with invertible diagonals.

We set $Z = Q$, $Z' = Q' \oplus O$ and $\tilde{Z} = Q \oplus O$, $\tilde{Z}' = Q'$. This gives the desired splittings $X \simeq Z \oplus Y \oplus Z'$ and $X \simeq \tilde{Z} \oplus Y \oplus \tilde{Z}'$.

Lemma 6.16. *Consider the setup of Construction 6.15 and the counterclockwise and clockwise splittings*

$$X \simeq Z \oplus Y \oplus Z' \quad \text{and} \quad X \simeq \tilde{Z} \oplus Y \oplus \tilde{Z}'\,.$$

Denote by e_1 the edge of $\mathcal{T}$ following the edge e in the counterclockwise direction and by e_2 the edge of $\mathcal{T}$ following the edge e in the clockwise direction.

(1) Suppose that $Y \simeq M_1$. Then any autoequivalence ϕ of $\mathrm{ev}_{e_1}(Z) \oplus \mathrm{ev}_{e_1}(Y) \oplus \mathrm{ev}_{e_1}(Z')$ given by a lower triangular matrix lifts to an autoequivalence of $Z \oplus Y \oplus Z'$ given by a lower triangular matrix.

(2) Suppose that $Y \simeq M_2$. Any autoequivalence ϕ of $\mathrm{ev}_{e_2}(\tilde{Z}) \oplus \mathrm{ev}_{e_2}(Y) \oplus \mathrm{ev}_{e_2}(\tilde{Z}')$ given by a lower triangular matrix lifts to an autoequivalence of $\tilde{Z} \oplus Y \oplus \tilde{Z}'$ given by a lower triangular matrix.

Proof. We only prove part (1), part (2) is analogous. By construction, we can find splittings $Z \simeq \bigoplus_{i_1} M_1 \oplus \bigoplus_{j_1} M_2$ and $Z' \simeq \bigoplus_{i_2} M_1 \oplus \bigoplus_{j_2} M_2 \oplus \bigoplus_{l} M_3$. We have $\mathrm{ev}_{e_1}(M_2) \simeq 0$ and $\mathrm{ev}_{e_1}(M_3) \simeq \mathrm{ev}_{e_1}(M_1) \simeq k[t_1^{\pm}]$. We further find $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_x, M_y) \xrightarrow{\mathrm{ev}_{e_1}} \mathrm{Mor}_{\mathcal{D}(k[t_1^{\pm}])}(k[t_1^{\pm}], k[t_1^{\pm}])$ to be an equivalence for $x = y = 1$, for $x = 3$ and $y = 1$ or for $x = y = 3$. It is thus clear that we can find a unique lift of ϕ, which restricts to the identity on $\bigoplus_{j_1} M_2 \oplus \bigoplus_{j_2} M_2$. □

By Lemma 6.16, the clockwise and counterclockwise splitting have the advantage that we can lift certain autoequivalences of their values at an edge to autoequivalences of the splitting. We use these autoequivalences to tweak the choices of local splittings in the proof of the geometrization Theorem 6.13, by replacing some splittings with their image under such an autoequivalence. After the necessary tweaks, we are able to glue the arising two-term splittings $Y \oplus (Z \oplus Z') \simeq X$ or $Y \oplus (\tilde{Z} \oplus \tilde{Z}') \simeq X$ to a splitting of global sections.

Proof of Theorem 6.13, part 1: existence of the decomposition. Given an edge e of $\mathcal{T}$ and $X \in \mathcal{C}_{\mathbf{S}}$, we denote by $\mathrm{ev}_e(X) \in \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e) \simeq \mathcal{D}(k[t_1^{\pm}])$ the value of the coCartesian section X at e. We note that the functor ev_e preserves compact objects, since it admits a colimit preserving right adjoint by [Chr25a, Lem. 4.3, Lem. 4.9, Prop. 4.23]. Thus $\mathrm{ev}_e(X)$ lies in $\mathcal{D}(k[t_1^{\pm}])^{\mathrm{perf}}$, meaning it is free of finite rank, using that $\mathrm{ho}\,\mathcal{D}(k[t_1^{\pm}]) \simeq \mathrm{Vect}_k$ is semi-simple.

Similarly, given a vertex v of $\mathcal{T}$, we denote by $\mathrm{res}_v(X)$ the restriction of the coCartesian section X to v and its three incident edges. The object $\mathrm{res}_v(X)$ can be considered as an object in $\mathcal{C}_{\mathbf{S}_v}$, with $\mathbf{S}_v$ a 3-gon. Note also that $\mathcal{C}_{\mathbf{S}_v} \simeq \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(v)$, with the equivalence given by evaluation at v. We show that if $X \neq 0$, then there is a splitting $X \simeq M_\gamma \oplus T$ for a matching datum $(\gamma, k[t_1^{\pm}])$. By a descending induction on the total rank of $\bigoplus_{e \in \mathcal{T}} \mathrm{ev}_e(X)$, the existence of the desired decomposition then follows.

Let $0 \neq X \in \mathcal{C}_{\mathbf{S}}$ be compact. Let e_0 be an external edge of $\mathcal{T}$ with $\mathrm{ev}_{e_0}(X) \neq 0$. If such an external edge does not exist, choose e_0 instead to be an internal edge with $\mathrm{ev}_{e_0}(X) \neq 0$. Choose any direct sum decomposition $B_0 \oplus A_0 \oplus B_0' \simeq \mathrm{ev}_{e_0}(X)$ with $A_0 \simeq k[t_1^{\pm}]$. Let v_1 be a trivalent vertex incident to e_0.

With this starting data, we iteratively make choices of edges e_i with incident vertices v_i and v_{i+1} and find splittings $B_i \oplus A_i \oplus B_i' \simeq \mathrm{ev}_{e_i}(X)$ and $T_i \oplus S_i \oplus T_i' \simeq \mathrm{res}_{v_i}(X)$ as follows.

Suppose the data for i has been chosen. Let $v_{i+1} \neq v_i$ be the other vertex incident to e_i. The summand $S_i \coloneqq Y$ appearing in both the clockwise and counterclockwise splittings of $\mathrm{res}_{v_{i+1}}(X)$ is of the form $Y \simeq M_\delta$, with δ a segment starting at the edge e_i and ending at another edge, called e_{i+1}. If e_{i+1} follows e_i in the clockwise direction, we use the clockwise splitting from Construction 6.15 arising from $B_i \oplus A_i \oplus B_i'$ to obtain a splitting $T_{i+1} \oplus S_{i+1} \oplus T_{i+1}'$ of $\mathrm{res}_{v_{i+1}}(X)$. If e_{i+1} instead follows e_i in the counterclockwise direction, we instead use the counterclockwise splitting. By Construction 6.15, we have an equivalence $\phi\colon \mathrm{ev}_{e_i}(T_{i+1}) \oplus \mathrm{ev}_{e_i}(S_{i+1}) \oplus \mathrm{ev}_{e_i}(T_{i+1}') \simeq \mathrm{ev}_{e_i}(X) \simeq B_i \oplus A_i \oplus B_i'$ which is given by a lower triangular matrix with invertible diagonal. By Lemma 6.16, we can find compatible autoequivalences of $T_j \oplus S_j \oplus T_j'$ for all $1 \leq j \leq i-1$ and of $B_j \oplus A_j \oplus B_j'$ for all $0 \leq j \leq i-1$, such that the composite at $B_i \oplus A_i \oplus B_i'$ with ϕ is a diagonal matrix. We redefine all $S_j, T_j, T_j', A_j, B_j, B_j'$ by their images under these autoequivalences. We then set $A_{i+1} = \mathrm{ev}_{e_{i+1}}(S_{i+1})$ and $B_{i+1} = \mathrm{ev}_{e_{i+1}}(T_{i+1})$, $B_{i+1}' = \mathrm{ev}_{e_{i+1}}(T_{i+1}')$.

We proceed with making these choices, until we come to a stop in one the following cases below. If e_0 is external, we always come to a stop in the following case, as otherwise one can find a contradiction to X being a coCartesian section.

Case 1) We have started at an external edge e_0 and arrive at an external edge e_N with $N \geq 1$.

In this case, the S_i's, with $1 \leq i \leq N$, glue to a global section $S \subset X$, satisfying $\mathrm{res}_v(S) = \bigoplus_{v_j = v} S_j$. Similarly, there is a global section $T \subset X$ satisfying $\mathrm{res}_v(T) = \bigcap_{v_j = v} T_j \oplus T_j'$ if there is at least one $1 \leq j \leq N$ with $v_j = v$ and $\mathrm{res}_v(T) = \mathrm{res}_v(X)$ if there are no j with $v_j = v$. The arising map $S \oplus T \to X$ restricts pointwise to an equivalence, and is hence also an equivalence of global sections. At each vertex v_i, we have by construction $S_i \simeq M_{\delta_i}$ for a segment δ_i at v_i. We compose these segments to an open pure matching curve γ. We find that $S \simeq M_\gamma$ is geometric. This gives the desired splitting $X \simeq M_\gamma \oplus T$.

Suppose now that we did not stop in case 1). We may then assume that $\mathrm{ev}_e(X) \simeq 0$ for all external edges e.

Case 2) There is a non-empty set $J \subset \{0, \dots, N-1\}$, such that $e_N = e_j$ for all $j \in J$ and $A_N \subset \langle A_j \rangle_{j \in J}$ lies in the submodule of $\mathrm{ev}_{e_N}(X)$ generated by the A_j's. Note that in this case J contains the element 0, as otherwise we again get a contradiction to X being coCartesian. We obtain a direct summand $S \subset X$ with complement T, satisfying $\mathrm{res}_v(S) = \bigoplus_{v_j = v,\, j<N} S_j$ and $\mathrm{res}_v(T) = \bigcap_{v_j = v,\, j<N} T_j \oplus T_j'$ or $\mathrm{res}_v(T) = \mathrm{res}_v(X)$ if there are no $j < N$, such that $v_j = v$.

Again, at each vertex v_i we have by construction $S_i \simeq M_{\delta_i}$ for a segment δ_i at v_i. We compose these segments to a closed curve γ. If γ is given by the composite of a identical closed curves γ', we have that the number N of segments of γ is a multiple of a and $e_i = e_{i+jN/a}$ for $1 \leq j \leq a-1$ and $0 \leq i \leq N/a$. We relabel $A_i = \bigoplus_{0 \leq j \leq a-1} A_{i+jN/a}$ and $S_i = \bigoplus_{0 \leq j \leq a-1} S_{i+jN/a}$ for $0 \leq i < N/a$. We have chosen γ' so that it is not again given by the composite of multiple identical closed curves. To extend γ' to a matching datum, we need to specify a rank and a monodromy datum.

We choose an ordered basis U_0 of $A_0 = \mathrm{ev}_{e_0}(S_1) \in \mathcal{D}(k[t_1^{\pm}])$ with cardinality a. Consider the basis U_1' of $\mathrm{res}_{v_1}(S_1)$ which is mapped under $S_1(v_1 \to e_0)$ to the chosen basis U_0. The image of U_1' under $S_1(v_1 \to e_1)$ defines an ordered basis U_1 of A_1. From this, we again find a basis U_2' of $\mathrm{res}_{v_2}(S_2)$. Proceeding in this way, performing these steps N/a times, we obtain a second ordered basis $U_{N/a}$ of $A_{N/a} = A_0$. Consider the Jordan normal form of the linear map, which maps $U_{N/a}$ to U_0. For each Jordan $b \times b$-block with eigenvalue $\lambda \neq 0$, we can equip γ with rank b and monodromy datum λ. With these choices of monodromy data, we finally have $S \simeq \bigoplus_{\mathrm{blocks}} M_\gamma$, where the sum runs over the Jordan blocks. We thus have the desired splitting $X \simeq \bigoplus_{\mathrm{blocks}} M_\gamma \oplus T$, concluding the proof. $\square$

Proof of Theorem 6.13, part 2: uniqueness of decomposition. Part 1 of the proof of Theorem 6.13 and the proof of Corollary 6.14 show that $\mathcal{C}_{\mathbf{S}}$ is Krull-Schmidt, or equivalently the homotopy 1-category $\mathrm{ho}\,\mathcal{C}_{\mathbf{S}}$ is Krull-Schmidt. The essential uniqueness of a decomposition into indecomposables in a Krull-Schmidt category is shown in [Kra15, Theorem 4.2]. It thus suffices to show that for two matching curves γ, γ', we have $M_\gamma \simeq M_{\gamma'}$ if and only if $\gamma \sim \gamma'$. The implication that $\gamma \sim \gamma'$ implies $M_\gamma \simeq M_{\gamma'}$ is straightforward. We thus suppose that $M_\gamma \simeq M_{\gamma'}$ and show $\gamma \sim \gamma'$. It is also straightforward to see that $M_\gamma \simeq M_{\gamma'}$ implies that γ and γ' are either both open or both closed.

Let us suppose that γ, γ' are both open. Then there exists an external edge e with $0 \not\simeq \mathrm{ev}_0(M_\gamma) \simeq \mathrm{ev}_0(M_{\gamma'})$, and so both curves must have an endpoint near e. We first suppose that both curves have a single endpoint near e and consider the case that they each have two at the end. Starting near e, the curves γ, γ' are composed of segments $\delta_1, \dots, \delta_m$ and $\delta_1', \dots, \delta_{m'}'$. Since $m + 1 = \mathrm{rk} \bigoplus_{f \in \mathcal{T}_1} \mathrm{ev}_f(M_\gamma) \in \mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}])$ and $m' + 1 = \mathrm{rk} \bigoplus_{f \in \mathcal{T}_1} \mathrm{ev}_f(M_\gamma) \in \mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}])$, we must have $m = m'$. Supposing that $\gamma \not\sim \gamma'$, there exists a smallest index $i \in [1, m]$, such that $\delta_i \neq \delta_i'$. Swapping γ, γ' if necessary, we can assume that there is a directed boundary intersection from δ_i to δ_i' in the marked triangle containing these. Stated differently, the means that γ turns right, whereas γ' turns lefts at the i-th segment. In this case, there is thus a directed boundary intersection from γ to γ' at e but not from γ' to γ. Using Theorem 6.8, this means that there is a morphism $M_\gamma \to M_{\gamma'}$ which induces an equivalence under ev_e but there is no morphism $M_{\gamma'} \to M_\gamma$ with this property. This is a contradiction to $M_\gamma \simeq M_{\gamma'}$. Thus $\gamma = \gamma'$.

Suppose next that both endpoints of γ and γ' lie near e and suppose that $\gamma \not\sim \gamma'$. We can order the four ends of γ, γ' by the order in which their endpoints induce endpoint intersections: how to order two endpoints of two distinct curves starting at e was explained in the previous paragraph. One considers the first pair of segments that differ and the first curve in the order is the curve that turns right in the triangle, whereas the second turns left. For there to be an equal number of directed boundary intersections from γ to γ', there must be one end of γ that precedes both ends of γ' in the order and one end of γ' that follows both ends of γ' in the order (or vice versa, swapping γ and γ'). But this means that no morphism $M_\gamma \to M_{\gamma'}$ can induce an equivalence under ev_e, since any directed boundary intersection from γ to γ' induces a morphism $\mathrm{ev}_e(M_\gamma) \simeq k[t_1^{\pm}]^{\oplus 2} \xrightarrow{(\mathrm{id},0)} k[t_1^{\pm}] \xrightarrow{(a,b)} k[t_1^{\pm}]^{\oplus 2} \simeq \mathrm{ev}_e(M_{\gamma'})$, where the parameters a, b correspond to the two endpoint intersections. We note that morphisms not arising from boundary intersections at e evaluate trivially under ev_e. Since $M_\gamma \simeq M_{\gamma'}$, this equivalence induces an equivalence under ev_e, which is a contradiction. Thus $\gamma \sim \gamma'$, concluding the argument in the case that γ, γ' are open.

If γ and γ' are closed, we distinguish the cases that the matching curves underlying γ and γ' are identical (up to reversal of orientation) or not. If they are identical, then the rank of γ and γ' must agree since the rank of $\bigoplus_{e \in \mathcal{T}_1} \mathrm{ev}_e(M_\gamma) \in \mathcal{D}^{\mathrm{perf}}(k))$ equals the number of segments of γ multiplied with the rank of γ. In this case, we have $\mathrm{rk}\,\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) = \mathrm{rk}\,\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma) - 2a$, see Theorem 6.8. It follows that $M_\gamma, M_{\gamma'}$ must also be distinct. The second case is that $\gamma \not\sim \gamma'$. We choose an internal edge e of $\mathcal{T}$ with $\mathrm{ev}_e(M_\gamma) \not\simeq 0$ and one of its two half edges h. We can cut γ or γ' at e at any single segment of theirs starting at e. The result is each an open curve composed of segments, going in the direction of the chosen halfedge. We can order the endpoints of these curves at e going in the direction of h as explained in the previous paragraphs. We consider the open curve composed of segments η that is maximal in this order. Swapping γ, γ' if necessary, we can assume that η arose from cutting open γ. To η we can associate a lax section M_η of $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$, see [Chr21, Constr. 4.13]. By [Chr21, Constr. 4.14], there is a morphism of lax sections $M_\eta \to M_{\gamma'}$ which extends to a coequalizer diagram. It has the property that it evaluates at e on the direct summands corresponding to the two endpoints of η to the morphism $(\mathrm{id}_{k[t_1^\pm]^{\oplus a}}, \mathcal{J}^{-1})\colon k[t_1^\pm]^{\oplus 2a} \to k[t_1^\pm]^{\oplus a}$ with $\mathcal{J}$ the Jordan matrix corresponding to the monodromy datum μ. The morphism object $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\eta, M_{\gamma'})$ is described in [Chr21, Rem. 5.12]. It roughly speaking consists of the usual intersections, together with directed intersections (which behave similar to boundary intersections) starting at e. Since η was chosen maximally, there are no directed intersection from M_η to $M_{\gamma'}$ at e. But this implies that the morphism $M_\eta \to M_\gamma \simeq M_{\gamma'}$ from the coequalizer diagram cannot exist, which is a contradiction. $\square$

7 The categorification of cluster algebras with coefficients

For the entire section, we fix a marked surface $\mathbf{S}$. We will further assume that the base field k is algebraically closed and satisfies $\mathrm{char}(k) \neq 2$.

We begin in the Sections 7.1 and 7.2 by recalling the definitions of the cluster algebra and of the commutative skein algebra associated with $\mathbf{S}$. In Section 7.3, we show that the cluster tilting objects in the exact ∞-/extriangulated cosingularity category $\mathcal{C}_{\mathbf{S}}$ are in bijection with the clusters of this cluster algebra. We also describe the endomorphism algebras of the relative cluster tilting objects in $\mathcal{C}_{\mathbf{S}}$ in terms of gentle algebras and discuss two simple examples. In Section 7.4, we describe a cluster character on $\mathcal{C}_{\mathbf{S}}$ with values in the commutative skein algebra.

7.1 Cluster algebras of marked surfaces

In the following we recall the definition of the cluster algebra associated to $\mathbf{S}$ with coefficients in the boundary arcs. These cluster algebras also have an alternative description via coordinates on the decorated Teichmüller space of the surface, called lambda lengths. These cluster algebras were studied in detail in Fomin-Thurston's beautiful paper [FT18] and correspond to the SL_2-case in the work of Fock-Goncharov [FG06].

For the conventions on cluster algebras, we follow [FWZ16, Chapter 3].

Definition 7.1. Let $\mathscr{F} = \mathbb{Q}(y_1, \dots, y_{m_1+m_2})$ be the field of rational functions in $m_1 + m_2$ variables. A (labeled) seed $\big(\mathbf{x}, \tilde{M}\big)$ in $\mathscr{F}$ consists of

- an $m_1 + m_2$-tuple $\mathbf{x} = (x_1, \dots, x_{m_1+m_2})$ in $\mathscr{F}$ forming a free generating set of $\mathscr{F}$ and
- an $(m_1+m_2) \times m_1$-matrix $\tilde{M}$, such that the upper $m_1 \times m_1$-matrix is skew-symmetric[5].

The tuple $\mathbf{x}$ is called a cluster and the elements $x_1, \dots, x_{m_1+m_2}$ are called cluster variables. The elements $x_{m_1+1}, \dots, x_{m_1+m_2}$ are also called the frozen cluster variables. The matrix $\tilde{M}$ is called the extended mutation matrix.

Definition 7.2. Let $\big(\mathbf{x}, \tilde{M}\big)$ be a seed and $l \in \{1, \dots, m_1\}$. The seed mutation at l is given by the seed $\big(\mathbf{x}', \mu_l(\tilde{M})\big)$ with

- $\mu_l(\tilde{M})_{i,j} = \begin{cases} -\tilde{M}_{i,j} & \text{if } i = l \text{ or } j = l \\ \tilde{M}_{i,j} + \tilde{M}_{i,l}\tilde{M}_{l,j} & \text{if } \tilde{M}_{i,l} > 0 \text{ and } \tilde{M}_{l,j} > 0 \\ \tilde{M}_{i,j} - \tilde{M}_{i,l}\tilde{M}_{l,j} & \text{if } \tilde{M}_{i,l} < 0 \text{ and } \tilde{M}_{l,j} < 0 \\ \tilde{M}_{i,j} & \text{else.} \end{cases}$

[5]More generally, one can also consider skew-symmetrizable matrices, see [FWZ16, Definition 3.1.1].

- $\mathbf{x}' = (x_1, \ldots, x_{l-1}, x'_l, x_{l+1}, \ldots, x_{m_1+m_2})$, where x'_l is determined by the cluster exchange relation

$$x'_l x_l = \prod_{j \text{ with } \tilde{M}_{j,l}>0} x_j^{\tilde{M}_{j,l}} + \prod_{j \text{ with } \tilde{M}_{j,l}<0} x_j^{-\tilde{M}_{j,l}} .$$

Definition 7.3. Let $(\mathbf{x}, \tilde{M})$ be a seed.

- The associated cluster algebra $\mathrm{CA} \subset \mathscr{F}$ is the $\mathbb{Q}$-subalgebra of $\mathscr{F}$ generated by all cluster variables in all seeds obtained from $(\mathbf{x}, \tilde{M})$ via iterated seed mutation.
- The associated upper cluster algebra $\mathrm{UCA} \subset \mathscr{F}$ is the $\mathbb{Q}$-subalgebra consisting of those elements which, for every cluster of CA, are Laurent polynomials in the cluster variables of that cluster.

Remark 7.4. The cluster algebra or upper cluster algebra associated to a seed only depends on the extended mutation matrix, up to isomorphism of $\mathbb{Q}$-algebras. We can thus speak of the cluster algebra or upper cluster algebra associated to an extended mutation matrix.

We proceed in Definition 7.5 with associating an extended mutation matrix to a choice of ideal triangulation I of $\mathbf{S}$ in the sense of Definition 6.5. Choosing a different ideal triangulation changes the extended mutation matrix by matrix mutations.

Definition 7.5 ([FST08, Definition 4.1], [FT18])**.** Let I be an ideal triangulation of $\mathbf{S}$ with m_1 interior arcs, labeled arbitrarily as $1, \ldots, m_1$, and m_2 boundary arcs, labeled arbitrarily as $m_1+1, \ldots, m_1+m_2$.

The extended signed adjacency matrix $\tilde{M}_I = \sum_\Delta M_\Delta$ of I is the $(m_1+m_2) \times m_1$-matrix given by the sum over all ideal triangles Δ of I of the $(m_1+m_2) \times m_1$-matrices defined by

$$(M_\Delta)_{i,j} = \begin{cases} 1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ with } i \text{ following } j \text{ in the counterclockwise direction,} \\ -1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ with } i \text{ following } j \text{ in the clockwise direction,} \\ 0 & \text{else.} \end{cases}$$

The upper $m_1 \times m_1$-matrix of $\tilde{M}_I$ is skew-symmetric and called the signed adjacency matrix.

Definition 7.6.

- Let $\mathbf{S}$ be an oriented marked surface with an ideal triangulation I. We define $\mathrm{CA}_\mathbf{S}$ to be the cluster algebra associated to the extended mutation matrix given by the extended signed adjacency matrix $\tilde{M}_I$ of I. Similarly, $\mathrm{UCA}_\mathbf{S}$ is defined as the associated upper cluster algebra.
- We define the algebra $\mathrm{CA}_\mathbf{S}^{\mathrm{loc}}$ as the localization of $\mathrm{CA}_\mathbf{S}$ at the frozen cluster variables $x_{m_1+1}, \ldots, x_{m_1+m_2}$, where m_1 is the number of interior arcs in I and m_2 is the number of boundary arcs in I.

Definition 7.7. Let γ, γ' be two arcs in $\mathbf{S}$. Suppose that γ, γ' have a crossing as in Figure 2. Then the Ptolemy relation is defined as $\gamma \cdot \gamma' = \gamma_1 \cdot \gamma_2 + \gamma_3 \cdot \gamma_4$.

Theorem 7.8. *The cluster variables of* $\mathrm{CA}_\mathbf{S}$ *are canonically in bijection with the arcs in* $\mathbf{S}$*. A set of cluster variables of* $\mathrm{CA}_\mathbf{S}$ *forms a cluster if and only if the corresponding arcs form an ideal triangulation of* $\mathbf{S}$*. The cluster exchange relations are the Ptolemy relations.*

Proof. This is [FT18, Theorem 8.6] specialized to the case that $\mathbf{S}$ has no punctures. □

7.2 Commutative skein algebras

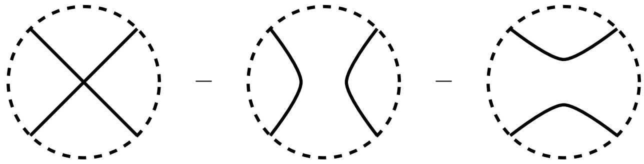
We proceed with defining the commutative $q = 1$ skein algebra $\mathrm{Sk}^1(\mathbf{S})$ of links in $\mathbf{S}$. As shown in [Mul16], this algebra embeds into the upper cluster algebra $\mathrm{UCA}_\mathbf{S}$.

Definition 7.9. A link is a homotopy class relative $\partial\mathbf{S}\backslash M$ of continuous maps $\gamma\colon U \to \mathbf{S}\backslash M$, with U a finite disjoint union of $[0,1]$'s and S^1's, satisfying properties 1. and 2. from Definition 6.1. We consider links up to reversal of orientation. We refer to the curves with domain $U = [0,1], S^1$ constituting a link as its components. There is an empty link with $U = \emptyset$. We denote by $\mathscr{L}(\mathbf{S})$ the set of all links in $\mathbf{S}$.

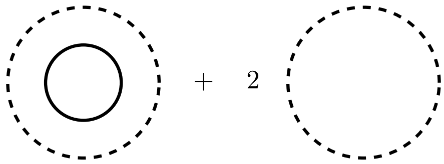
The $\mathbb{Q}$-vector space $\mathbb{Q}^{\mathcal{L}(\mathbf{S})}$ inherits the structure of a commutative $\mathbb{Q}$-algebra, by defining the product of two links to be the union of the two links and extending this product $\mathbb{Q}$-bilinearly.

Definition 7.10. We define the $q = 1$ skein algebra $\mathrm{Sk}^1(\mathbf{S})$ as the quotient of the $\mathbb{Q}$-algebra $\mathbb{Q}^{\mathcal{L}(\mathbf{S})}$ by the ideal generated by the following elements.

1) The $q = 1$ Kauffman skein relation.



2) The value of the unknot.



3) Any component with domain $[0, 1]$ which is homotopic relative endpoints to a subset of $\partial\mathbf{S}\backslash M$.

The relations in 1) and 2) are understood to describe local relations inside a small disc (indicated by the dotted circle) in $\mathbf{S}$, applicable to any subset of components of a link. The depicted curves are identical outside the small disc.

Remark 7.11. The $q = 1$ Kauffman skein relation recover the cluster exchange relations given by the Ptolemy relation, see Definition 7.7.

We remark that Definition 7.10 is equivalent to the definition of the q-skein algebra given by Muller in [Mul16], with q set to 1, as can be seen by using Remarks 3.4 and 3.6 in [Mul16].

Definition 7.12. We define the localized $q = 1$ skein algebra $\mathrm{Sk}^{1,\mathrm{loc}}(\mathbf{S})$ as the localization of $\mathrm{Sk}^1(\mathbf{S})$ at the set of boundary arcs.

Theorem 7.13 ([Mul16])**.** *There exist injective morphisms of $\mathbb{Q}$-algebras*

$$\mathrm{CA}^{\mathrm{loc}}_{\mathbf{S}} \hookrightarrow \mathrm{Sk}^{1,\mathrm{loc}}(\mathbf{S}) \hookrightarrow \mathrm{UCA}_{\mathbf{S}}\,. \tag{14}$$

If $\mathbf{S}$ additionally has at least two marked points, then the maps in (14) are equivalences of $\mathbb{Q}$-algebras.

7.3 Classification of cluster tilting objects

We choose a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$. Consider the perverse schober $\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}$ from Lemma 4.3 with its $k[t_2^{\pm}]$-linear smooth and proper ∞-category of global sections equivalent to the cosingularity category $\mathcal{C}_{\mathbf{S}}$. We let G be the right adjoint of the adjunction

$$F \coloneqq \partial\mathcal{F}^{\mathrm{clst}}_{\mathcal{T}} \colon \prod_{e\in\mathcal{T}_1^\partial} \mathcal{F}^{\mathrm{clst}}_{\mathcal{T}}(e) \longleftrightarrow \mathcal{C}_{\mathbf{S}} : G \coloneqq \prod_{e\in\mathcal{T}_1^\partial} \mathrm{ev}_e$$

from Definition 3.11. Since $\mathrm{char}(k) \neq 2$, the functor G admits a 2-Calabi-Yau structure, see Theorem 4.17 and the adjunction $F \dashv G$ is spherical by Theorem 3.13. By Section 5.3, we obtain a Frobenius, 2-Calabi-Yau extriangulated category $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ arising from of a Frobenius exact ∞-structure on $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$.

Theorem 7.14. *Let k be an algebraically closed field with* $\mathrm{char}(k) \neq 2$.

i) Let γ be a matching curve in $\mathbf{S}$. Then M_γ is rigid in $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ if and only if γ is an arc.

ii) Consider a collection I of distinct arcs in $\mathbf{S}$. Then $\bigoplus_{\gamma\in I} M_\gamma$ is a cluster tilting object in $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ if and only if I is an ideal triangulation of $\mathbf{S}$.

We prove Theorem 7.14 further below.

Corollary 7.15. *There are canonical bijections between the sets of the following objects.*

- *Clusters of the cluster algebra with coefficients $\mathrm{CA}_{\mathbf{S}}$ of $\mathbf{S}$.*
- *Ideal triangulations of $\mathbf{S}$.*
- *Cluster tilting objects in $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$ up to equivalence.*

Proof. Combining the geometrization Theorem 6.13 and Theorem 7.14, we obtain that there is a bijection between the sets of equivalence classes of cluster tilting objects and ideal triangulations of $\mathbf{S}$. The bijection between clusters and ideal triangulation is Theorem 7.8. □

Lemma 7.16. *(1) Let γ, γ' be two distinct matching curves in $\mathbf{S}$. Then*

$$\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$$

consists of those direct summands identified in Theorem 6.8 which correspond to crossings of γ and γ'.

(2) Let γ be an open matching curve in $\mathbf{S}$. Then

$$\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma}) \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma})$$

consists of those direct summands identified in Theorem 6.9 which corresponding to self-crossings.

(3) Let γ be a closed matching curve in $\mathbf{S}$. Then

$$\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma}) = \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma})\,.$$

Proof. Inspecting the construction of the direct summands of $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$ corresponding to the different types of intersections in [Chr21], one finds the following.

- Boundary intersections give rise to morphisms which evaluate non-trivially at an external edge of $\mathcal{T}$, i.e. do not get mapped by G to zero.
- Crossings give rise to morphisms which restrict to zero on all external edges of $\mathcal{T}$ and thus define direct summands lying in $\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$.
- If γ is open, then the direct summand $k[t_1^{\pm}] \subset \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma)$ does not evaluate to zero under G.
- If γ is closed, then $G(M_\gamma) \simeq 0$ and hence $\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma) = \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma)$.

This shows the Lemma. □

Proof of Theorem 7.14. We begin with part i). If γ is a closed matching curve, then M_γ is not rigid because there is always a direct summand $k[t_1^{\pm}] \subset \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma) = \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma)$. We thus assume that γ is open. Theorem 6.9 and Lemma 7.16 imply that M_γ is rigid if and only if γ has no self crossings, meaning that γ is an arc, showing part i).

Using Theorem 6.8 and Lemma 7.16, we find that given two arcs γ, γ', we have an isomorphism $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) \simeq 0$ if and only if γ and γ' are compatible in the sense of Definition 6.5. By part i) and the maximality of an ideal triangulation, we find that any basic, maximal rigid object is of the form $\bigoplus_{\gamma\in I} M_\gamma$ for an ideal triangulation I and conversely any ideal triangulation I gives rise to such a basic, maximal rigid object. Since any cluster tilting object is basic and maximal rigid, it remains to verify that

$$\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(\bigoplus_{\gamma\in I} M_\gamma, M_{\gamma'}) \not\simeq 0 \tag{15}$$

for I an ideal triangulation and $\gamma' \notin I$ a matching curve.

Let thus I be an ideal triangulation. Then I decomposes $\mathbf{S}$ into triangles with edges the arcs in I. If a matching curve γ' crosses an edge of one of these triangles, we find a direct summand $k[t_1^{\pm}] \subset \mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$, showing (15). If γ' does not cross any arcs in I, then γ' is contained in an ideal triangle and hence already in I. We thus obtain that (15) holds for all $\gamma' \notin I$, showing that $\bigoplus_{\gamma \in I} M_\gamma$ is cluster tilting. $\square$

Recall that given a trivalent spanning graph $\mathcal{T}$ of $\mathbf{S}$, we denote the associated relative Ginzburg algebra by $\mathscr{G}_{\mathcal{T}}$.

Proposition 7.17. *Consider an ideal triangulation I of $\mathbf{S}$ with dual trivalent spanning graph $\mathcal{T}$ and let $X = \bigoplus_{\gamma \in I} M_\gamma$ be the associated cluster tilting object.*

(1) The Jacobian algebra $\mathscr{J}_{\mathcal{T}} \coloneqq \mathrm{H}_0(\mathscr{G}_{\mathcal{T}})$ is a gentle algebra.

(2) There exists an isomorphism of dg-algebras (with vanishing differentials)

$$\mathrm{H}_* \operatorname{Mor}_{\mathcal{C}_{\mathbf{S}}}(X, X) \simeq \mathrm{H}_0(\mathscr{G}_{\mathcal{T}}) \otimes_k k[t_1^{\pm}]\,.$$

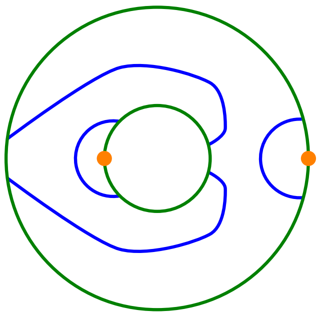
In particular, the discrete endomorphism algebra $\mathrm{Ext}^0_{\mathcal{C}_{\mathbf{S}}}(X, X)$ of X is a gentle algebra.

Proof. Part (1) is observed in [Chr21, Section 5.5]. Part (2) follows from Theorems 6.8 and 6.9 as in the proof of [Chr21, Prop. 5.18]. $\square$

Remark 7.18. The marked surface associated to the Jacobian gentle algebra $\mathrm{H}_0(\mathscr{G}_{\mathcal{T}})$ (in the sense of [OPS18, Def 1.10]) is given by the complement in $\mathbf{S}$ of the set of vertices of the trivalent spanning graph $\mathcal{T}$. This gentle algebra is finite dimensional but not smooth.

The Jacobian gentle algebra of the 4-gon is described in Example 1. Another example is as follows:

Example 7.19. Consider the annulus $\mathbf{S}$ with a marked point on each boundary component and an ideal triangulation depicted as follows.



The Jacobian gentle algebra is given by the quotient of the path algebra of the quiver

$$\begin{tikzcd}
 & 3 \arrow[dl, "b"'] & \\
1 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "a'"'] & & 2 \arrow[ul, "b"'] \arrow[dl, "b'"] \\
 & 3' \arrow[ul, "c'"] &
\end{tikzcd}$$

by the ideal $(ba, cb, ac, b'a', c'b', a'c')$. This is a relative version of the Kronecker quiver. While the mapping class group of the 4-gon in Example 1 is trivial, the mapping class group of the annulus $\mathbf{S}$ is given by $\mathbb{Z}$. The generator $1 \in \mathbb{Z}$ corresponds to a diffeomorphism rotating one boundary circle by one full rotation and fixing the other boundary circle. Under the action of an element α of the mapping class group, an object $M_\gamma \in \mathcal{C}_{\mathbf{S}}$ is mapped to $M_{\alpha \circ \gamma}$.

7.4 A cluster character on $\mathcal{C}_{\mathbf{S}}$

We begin with the notion of a cluster character on an extriangulated category, generalizing the notion of a cluster character of [Pal08].

Definition 7.20. Let $(C, \mathbb{E}, \mathfrak{s})$ be a k-linear extriangulated category. We denote by $\mathrm{obj}(C)$ the set of equivalence classes of objects in C. A cluster character χ on C with values in a commutative ring R is a map

$$\chi\colon\ \mathrm{obj}(C) \to R$$

such that for all $X, Y \in \mathcal{C}$ the following holds.

- $\chi(X) = \chi(Y)$ if $X \simeq Y$.
- $\chi(X \oplus Y) = \chi(X)\chi(Y)$.
- $\chi(X)\chi(Y) = \chi(B) + \chi(B')$ if $\dim_k \mathbb{E}(X, Y) = \dim_k \mathbb{E}(Y, X) = 1$ and $B, B' \in C$ are the middle terms of the corresponding non-split extensions of X by Y and Y by X.

The last property is called the cluster multiplication formula.

Given a matching curve γ in $\mathbf{S}$, we consider the underlying curve as a link in $\mathbf{S}$ with a single component, denoted $l(\gamma)$.

Theorem 7.21. *Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$. Consider the map*

$$\chi\colon\ \mathrm{obj}(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}) \to \mathrm{Sk}^1_{\mathbf{S}}$$

determined by

- $\chi(A) = \chi(B)$ *if* $A \simeq B$
- $\chi(A \oplus B) = \chi(A)\chi(B)$
- $\chi(M_\gamma) = l(\gamma)$ *for any matching curve* γ *in* $\mathbf{S}$.

Then χ is a cluster character on $(\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}, \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}, \mathfrak{s})$.

Remark 7.22. Composing χ with $\mathrm{Sk}^1_{\mathbf{S}} \hookrightarrow \mathrm{Sk}^{1,\mathrm{loc}}_{\mathbf{S}} \hookrightarrow \mathrm{UCA}_{\mathbf{S}}$ defines a cluster character to the upper cluster algebra of $\mathbf{S}$. If $\mathbf{S}$ has at least two marked points, the upper cluster algebra is equivalent to the cluster algebra with coefficients $\mathrm{CA}^{\mathrm{loc}}_{\mathbf{S}}$ localized at the boundary arcs, see Theorem 7.13.

Proof of Theorem 7.21. The geometrization Theorem 6.13 shows that χ is well-defined. It is also clear that χ satisfies all parts of Definition 7.20, except for the cluster multiplication formula. We thus consider two matching curves γ, γ' in $\mathbf{S}$, which satisfy $\dim_k \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) = 1$ and let $B, B' \in \mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ be the corresponding extensions. Suppose that γ and γ' are distinct. This allows the case that the underlying curves are closed and identical but the monodromy data differ. By Theorem 6.8, $\dim_k \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) = 1$ implies that γ, γ' have a single crossing. The cluster multiplication formula $\chi(M_\gamma)\chi(M_{\gamma'}) = \chi(B) + \chi(B')$ thus follows from Proposition 6.10, see also Remark 6.11.

If $\gamma = \gamma'$ and γ, each self crossing of γ gives rise to two crossings of γ, γ'. Thus, if γ has a self-crossing, the condition $\dim_k \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) = 1$ cannot be satisfied. If γ is open without a self-crossing, then $\dim_k \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) = 0$. We thus suppose that γ is closed without a self-crossing. If the monodromy data of γ, γ' differ, then again $\dim_k \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) = 0$. We thus suppose, that the monodromy data of γ and γ' agree. Denote by a and a' the ranks of γ and γ'. We show that $\dim_k \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$ is always an even number and hence not equal to 1. The global section M_γ arises as a coequalizer, see (13). It follows that $\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$ is equivalent to the fiber of some morphism

$$k[t_1^{\pm}]^{\oplus aa'} \longrightarrow k[t_1^{\pm}]^{\oplus aa'}(\gamma, \gamma')\,. \tag{16}$$

Inspecting the construction of the morphism (16), one finds that it lies in the image of the functor $\zeta^*\colon \mathcal{D}(k[t_1^{\pm}]) \to \mathcal{D}(k[t_2^{\pm}])$. Using that every morphism in $\mathcal{D}(k[t_1^{\pm}])$ splits into a direct sum of an equivalence and a zero morphism, it follows that $\mathrm{Mor}^{\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) = \mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'})$ is free of even rank in $\mathcal{D}(k[t_1^{\pm}])$, which implies that $\dim_k \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma)$ is indeed an even number. This concludes the case distinction and the proof. $\square$

Remark 7.23. The proof of Theorem 7.21 shows that there is some flexibility in the formula for the cluster character. In fact, it appears that one can assign arbitrary values to the closed matching curves of rank $a \geq 2$.

8 The equivalence with the Higgs category and further results

In Section 8.1, we recall the construction of the Higgs category [Wu23] and show that the 1-periodic topological Fukaya category $\mathcal{C}^{c}_{\mathbf{S}}$ is equivalent as an exact ∞-category to the Higgs category. From this, it follows that the (homotopy category of the) stable category of $\mathcal{C}^{c}_{\mathbf{S}}$ recovers the standard triangulated 2-Calabi–Yau cluster category of the marked surface $\mathbf{S}$. In Section 8.2, we observe that the mapping class group action on $\mathcal{C}_{\mathbf{S}}$ induces a mapping class group action on this triangulated 2-Calabi–Yau cluster category. Finally, in Section 8.3, we describe the generalized n-cluster categories arising from relative higher Ginzburg algebras associated with n-angulated marked surfaces.

8.1 The Higgs category

In [Wu23], Yilin Wu describes a generalization of Amiot's construction of the generalized cluster category, that is different to the approach of this paper and considers a more general setup. Given the input of suitable relative left 3-Calabi–Yau dg-algebra, the result is a 2-Calabi–Yau Frobenius extriangulated category equipped with cluster tilting objects. In the following, we recall this construction in the ∞-categorical setting and in the special case where the relative 3-Calabi–Yau dg-algebra is the relative Ginzburg algebras associated with an ice quiver with potential and show the compatibility with the approach of this paper if the ice quiver with potential comes from a triangulated marked surface.

Let (Q,F) be a finite ice quiver, meaning that Q is a finite quiver and $F \subset Q$ a (not necessarily full) sub-quiver. Let W be a potential for Q. Associated with this is the relative Ginzburg algebra $\mathscr{G}(Q,F,W)$ and the Ginzburg dg-functor [Wu23]

$$\mathbf{Gi}\colon \Pi_2(kF) \to \mathscr{G}(Q,F,W)\,,$$

where $\Pi_2(kF)$ denotes the 2-Calabi–Yau completion [Kel11] of the path algebra of F. The Ginzburg functor $\mathbf{Gi}$ is equivalent to the deformed relative Calabi–Yau completion of $kF \to kQ$ and thus admits a left 3-Calabi–Yau structure [Yeu16, BCS20]. We will assume that the relative Jacobian algebra $H_0(\mathscr{G}(Q,F,W))$ is finite dimensional and refer to [KW23] for a discussion of Higgs categories in the Jacobi-infinite case.

Pulling back along $\mathbf{Gi}$ yields the functor

$$\mathbf{Gi}^*\colon \mathcal{D}(\mathscr{G}(Q,F,W)) \to \mathcal{D}(\Pi_2(kF))\,.$$

Its left adjoint is denoted $\mathbf{Gi}_!$ and given by tensoring with the bimodule $\bigoplus_{x\in F_0} P_x$ given by the sum of the indecomposable direct summands of $\mathscr{G}(Q,F,W)$ associated with the frozen vertices.

We let $\mathcal{D}^{\mathrm{fin}}_F(\mathscr{G}(Q,F,W)) \subset \mathcal{D}^{\mathrm{fin}}(\mathscr{G}(Q,F,W))$ be the kernel, i.e. the fiber, of the functor $\mathbf{Gi}^*$. The relative cluster (∞-)category is defined as the Verdier quotient

$$\mathcal{C}^{\mathrm{rel}}(Q,F,W) \coloneqq \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))/\mathcal{D}^{\mathrm{fin}}_F(\mathscr{G}(Q,F,W))\,.$$

To define the Higgs (∞-)category, we must first recall the definition of the relative fundamental domain $\mathcal{F}^{\mathrm{rel}} \subset \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))$. Consider the set $\mathcal{P}$ of objects in $\mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))$ given by the indecomposable direct summands of $\mathscr{G}(Q,F,W)$ associated with the frozen vertices of Q. Note that $\mathrm{add}(\mathcal{P})$ is the essential image $\mathbf{Gi}_!(\mathrm{add}(kF))$ of $\mathrm{add}(kF)$ under the functor $\mathbf{Gi}_!$.

Definition 8.1. The relative fundamental domain $\mathcal{F}^{\mathrm{rel}} \subset \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))$ is the full subcategory spanned by objects X satisfying that

1) X fits into a fiber and cofiber sequence $M_1 \to M_0 \to X$ with M_0, M_1 lying in the additive closure $\operatorname{add}(\mathscr{G}(Q,F,W))$ and

2) $\operatorname{Ext}^i(P,X) \simeq \operatorname{Ext}^i(X,P) \simeq 0$ for all $P \in \mathcal{P}$ and $i > 0$.

Lemma 8.2. *Suppose that the functor* $\mathbf{Gi}^*$ *admits a colimit preserving right adjoint* $\mathbf{Gi}_*$ *satisfying that* $\operatorname{add}(\mathcal{P}) = \mathbf{Gi}_*(\operatorname{add}(\Pi_2(kF)))$*. Then condition 2) in Definition 8.1 is equivalent to* X *satisfying* $\mathbf{Gi}^*(X) \in \operatorname{add}(\Pi_2(kF))$*.*

Note that the condition that $\mathbf{Gi}_*$ preserves colimits is equivalent to $\mathbf{Gi}^*$ preserving compact objects, i.e. perfect modules.

Proof of Lemma 8.2. Let $X \in \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))$. Let $P \simeq \mathbf{Gi}_!(Q) \simeq \mathbf{Gi}_*(Q') \in \operatorname{add}(\mathcal{P})$ with $Q, Q' \in \operatorname{add}(\Pi_2(kF))$. We have

$$\operatorname{Ext}^i_{\mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))}(P,X) \simeq \operatorname{Ext}^i_{\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))}(Q, \mathbf{Gi}^*(X))$$

and

$$\operatorname{Ext}^i_{\mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))}(X,P) \simeq \operatorname{Ext}^i_{\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))}(\mathbf{Gi}^*(X), Q')\,.$$

Since the dg-algebra $\Pi_2(kF)$ is concentrated in positive degrees (in the homological grading convention), it defines a silting object in $\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))$. It follows that $\mathrm{Gi}^*(X) \in \operatorname{add}(\Pi_2(kF))$ holds if and only if

$$\operatorname{Ext}^i_{\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))}(\mathrm{Gi}^*(X), \Pi_2(kF)) \simeq \operatorname{Ext}^i_{\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))}(\Pi_2(kF), \mathrm{Gi}^*(X)) \simeq 0$$

for all $i > 0$, see [MSSS13, Thm. 5.5] or also [IY18, Lem. 2.5, Prop. 2.8], [KN13, Thm. 10.2.2]. By the above, this is satisfied if and only if $\operatorname{Ext}^i(P,X) \simeq \operatorname{Ext}^i(X,P) \simeq 0$ for all $P \in \mathcal{P}$ and $i > 0$. $\square$

The Higgs category $\mathcal{H}(Q,F,W) \subset \mathcal{C}^{\mathrm{rel}}(Q,F,W)$ is defined as the image of $\mathcal{F}^{\mathrm{rel}}$ under the quotient functor $\mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W)) \to \mathcal{C}^{\mathrm{rel}}(Q,F,W)$. The arising functor $\phi\colon \mathcal{F}^{\mathrm{rel}} \to \mathcal{H}(Q,F,W)$ is furthermore fully faithful on the level of homotopy categories [Wu23, Prop. 5.20]. The Higgs category is an extension closed subcategory of $\mathcal{C}^{\mathrm{rel}}(Q,F,W)$ and thus inherits the structure of an exact ∞-category. The homotopy category $\operatorname{ho}\mathcal{H}(Q,F,W)$ is therefore extriangulated. Furthermore, the image of $\mathscr{G}(Q,F,W)$ in $\mathcal{H}(Q,F,W)$ is a cluster tilting object.

We next discuss under the assumptions of Lemma 8.2 the relation of the Higgs category with the cosingularity category

$$\mathcal{C}(Q,F,W) \coloneqq \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W))/\mathcal{D}^{\mathrm{fin}}(\mathscr{G}(Q,F,W))\,.$$

We equip $\mathcal{C}(Q,F,W)$ with the exact ∞-structure obtained from pulling back as in Example 5.8 the split exact structure on $\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))/\mathcal{D}^{\mathrm{fin}}(\Pi_2(kF))$ along the functor induced by $\mathbf{Gi}^*$ (note that $\mathbf{Gi}^*$ automatically preserves finite modules and we have assumed that it preserves perfect modules):

$$\widetilde{\mathbf{Gi}}^*\colon \mathcal{C}(Q,F,W) \longrightarrow \mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))/\mathcal{D}^{\mathrm{fin}}(\Pi_2(kF))\,.$$

Composing the inclusion $\mathcal{H}(Q,F,W) \subset \mathcal{C}^{\mathrm{rel}}(Q,F,W)$ with the quotient functor $\mathcal{C}^{\mathrm{rel}}(Q,F,W) \to \mathcal{C}(Q,F,W)$ to the cosingularity category yields the functor

$$\tau\colon \mathcal{H}(Q,F,W) \to \mathcal{C}(Q,F,W)\,. \tag{17}$$

Proposition 8.3. *Suppose that the functor* $\mathbf{Gi}^*$ *admits a colimit preserving right adjoint* $\mathbf{Gi}_*$ *satisfying that* $\operatorname{add}(\mathcal{P}) = \mathbf{Gi}_*(\operatorname{add}(\Pi_2(kF)))$*. Then the functor* τ *is an exact functor between exact* ∞*-categories and thus also gives rise to an extriangulated functor between the extriangulated homotopy categories.*

Proof of Proposition 8.3. Since zero objects, inflations, deflations and exact sequences are detected on the level of the extriangulated homotopy categories, the functor τ can be lifted to an exact functor between exact ∞-categories if and only if $\operatorname{ho}\tau$ can be lifted to an extriangulated functor between the extriangulated homotopy categories. Lifting $\operatorname{ho}\tau\colon \operatorname{ho}\mathcal{H}(Q,F,W) \to \operatorname{ho}\mathcal{C}(Q,F,W)$ to an extriangulated functor amounts to specifying a natural transformation between functors $\operatorname{ho}\mathcal{H}(Q,F,W)^{\mathrm{op}} \times \operatorname{ho}\mathcal{H}(Q,F,W) \to \mathrm{Vect}_k$

$$\mathbb{E}_{\operatorname{ho}\mathcal{H}(Q,F,W)}(\text{-},\text{-}) \longrightarrow \mathbb{E}_{\operatorname{ho}\mathcal{C}(Q,F,W)}(\tau(\text{-}),\tau(\text{-}))\,,$$

such that τ applied to the realization $\mathfrak{s}(\alpha)$ of an extension α yields the realization $\mathfrak{s}(\tau(\alpha))$.

The fiber and cofiber sequence preserving functor τ induces a natural transformation $\operatorname{Ext}^1_{\mathcal{H}(Q,F,W)}(\text{-},\text{-}) \to \operatorname{Ext}^1_{\mathcal{C}(Q,F,W)}(\tau(\text{-}),\tau(\text{-}))$. Since $\mathbb{E}_{\operatorname{ho}\mathcal{H}(Q,F,W)} = \operatorname{Ext}^1_{\operatorname{ho}\mathcal{H}(Q,F,W)}$ and

$$\mathbb{E}_{\operatorname{ho}\mathcal{C}(Q,F,W)} = \ker(\widetilde{\mathbf{Gi}}^*|_{\operatorname{Ext}^1}) \subset \operatorname{Ext}^1_{\operatorname{ho}\mathcal{C}(Q,F,W)}$$

it suffices to show that $\widetilde{\mathbf{Gi}}^* \circ \tau$ vanishes on extensions to obtain the desired natural transformation.

Denote by $(\bar{Q},\bar{W})$ the (non-ice) quiver with potential obtained from (Q,F,W) by removing F from Q and all cycles from W which contain frozen arrows. Let $\mathscr{G}(\bar{Q},\bar{W})$ be the corresponding non-relative Ginzburg algebra and $\mathcal{C}(\bar{Q},\bar{W}) = \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(\bar{Q},\bar{W}))/\mathcal{D}^{\mathrm{fin}}(\mathscr{G}(\bar{Q},\bar{W}))$ the (∞-categorical version of the) cluster category. There is a cofiber sequence of compactly generated stable ∞-categories, see [Wu23, Prop. 7.8],

$$\mathcal{D}^{\mathrm{perf}}(\Pi_2(F)) \longrightarrow \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(Q,F,W)) \xrightarrow{\pi} \mathcal{D}^{\mathrm{perf}}(\mathscr{G}(\bar{Q},\bar{W}))\,.$$

The functor π induces the functor $\bar{\pi}\colon \mathcal{C}^{\mathrm{rel}}(Q,F,W) \to \mathcal{C}(\bar{Q},\bar{W})$.

Let $X_1, X_2 \in \mathcal{H}(Q,F,W)$ and $Y_1, Y_2 \in \mathcal{F}^{\mathrm{rel}}$ denote the images of X_1, X_2 under the equivalence $\operatorname{ho}\mathcal{F}^{\mathrm{rel}} \simeq \operatorname{ho}\mathcal{H}(Q,F,W)$. Let further $Y_1' = \tau^{\mathrm{rel}}_{\leq -1}(Y_1)$ be the relative truncation of Y_1, see [Wu23, Def. 4.9]. There is a morphism $Y_1' \to Y_1$, whose cofiber lies in $\mathcal{D}^{\mathrm{fin}}_F(\mathscr{G}(Q,F,W))$, see loc. cit.. The image of the morphism in $\mathcal{C}^{\mathrm{rel}}(Q,F,W)$ is thus equivalence. We have the following equivalences,

$$\begin{aligned}
\operatorname{Ext}^1_{\mathcal{H}(Q,F,W)}(X_1,X_2) &\simeq \operatorname{Ext}^1_{\mathcal{C}(\bar{Q},\bar{W})}(\bar{\pi}(X_1),\bar{\pi}(X_2)) \\
&\simeq \operatorname{Ext}^1_{\mathcal{D}(\mathscr{G}(\bar{Q},\bar{W}))}(\pi(Y_1'),\pi(Y_2)) \\
&\simeq \operatorname{Ext}^1_{\mathcal{D}(\mathscr{G}(Q,F,W)}(Y_1',Y_2)
\end{aligned}$$

where the first equivalence is [Wu23, Prop. 5.36], the second equivalence is shown in step 2 in the proof of Proposition 2.12 in [Ami09], using that $\pi(\tau^{\mathrm{rel}}_{\leq -1}(Y_1)) \simeq \tau_{\leq -1}\pi(Y_1)$, and the last equivalence is shown in [Wu23, Lem. 5.16, Lem. 5.29].

Any extension $\alpha\colon X_1 \to X_2[1]$ in $\mathcal{H}(Q,F,W)$ thus lifts to an extension $\alpha'\colon Y_1' \to Y_2[1]$ in $\mathcal{D}(\mathscr{G}(Q,F,W))$. We note that $\widetilde{\mathbf{Gi}}^* \circ \tau(\alpha)$ is equivalent to the image under the quotient functor $\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF)) \to \mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))/\mathcal{D}^{\mathrm{fin}}(\Pi_2(kF))$ of $\mathbf{Gi}^*(\alpha')$. Since $\mathbf{Gi}^*(Y_1'), \mathbf{Gi}^*(Y_2) \in \operatorname{add}(\Pi_2(kF))$ and $\Pi_2(kF)$ is rigid, we find that $\mathbf{Gi}^*(\alpha')$ must vanish. It follows that $\widetilde{\mathbf{Gi}}^* \circ \tau(\alpha) \simeq 0$, as desired. $\square$

The functor τ considered in Proposition 8.3 is not for every choice of ice quiver with potential an equivalence of extriangulated categories and the cosingularity category $\mathcal{C}(Q,F,W)$ does not necessarily categorify the corresponding cluster algebra with coefficients. This is because $\mathcal{C}(Q,F,W)$ may vanish for an arbitrary choice of ice quiver with potential. This is the case for instance if $\mathscr{G}(Q,F,W)$ is Jacobi-finite and concentrated in degree 0, see [Wu23] for examples. For a basic example where $\mathcal{C}(Q,F,W)$ vanishes, see Example 8.8.

Finally, we consider the specific choice of ice quiver with potential $(Q,F,W) = (Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}})$ associated with a marked surface $\mathbf{S}$ with an ideal triangulation I dual to a trivalent spanning graph $\mathcal{T}$. The vertices of $Q_{\mathcal{T}}$ are given by the arcs in I and the arrows are obtained by inscribing clockwise 3-cycles into the faces of the ideal triangulation. The frozen part $F_{\mathcal{T}}$ consists of the vertices given by boundary arcs. The potential $W_{\mathcal{T}}$ is the sum of the 3-cycles inscribed into the faces. The corresponding non-frozen quiver with potential $(\bar{Q}, \bar{W})$ was defined in [LF09]. We note that there is an isomorphisms of dg-algebras $\mathscr{G}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}}) \simeq \mathscr{G}_{\mathcal{T}}$ and thus an equivalence $\mathcal{C}(Q, F, W) \simeq \mathcal{C}_{\mathbf{S}}^{c}$ with the subcategory of compact objects of the (Ind-complete) 1-periodic topological Fukaya $\mathcal{C}_{\mathbf{S}}$ of $\mathbf{S}$.

Theorem 8.4. *The functor*

$$\tau\colon \mathcal{H}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}}) \to \mathcal{C}_{\mathbf{S}}^{c}$$

is an equivalence of exact ∞-categories. In particular, it induces an equivalence between the extriangulated homotopy categories.

Proof. The conditions of Proposition 8.3 are satisfied by Remark 8.5 below. We first show that $\mathrm{ho}\,\tau$ is an equivalence of extriangulated homotopy categories using Lemma 8.6 below.

Under the quotient functors $\mathcal{D}(\mathscr{G}_{\mathcal{T}}) \to \mathcal{C}^{\mathrm{rel}}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}}), \mathcal{C}_{\mathbf{S}}$, the relative Ginzburg algebra is mapped to cluster tilting objects $T \in \mathcal{H}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}})$ and $T' \simeq \bigoplus_{\gamma\in I} M_\gamma \in \mathcal{C}_{\mathbf{S}}$, respectively. The Homs assemble into an apparent commutative diagram, where γ is induced by the functor τ.

$$\begin{tikzcd}
\mathrm{Ext}^0_{\mathcal{D}(\mathscr{G}_{\mathcal{T}})}(\mathscr{G}_{\mathcal{T}}, \mathscr{G}_{\mathcal{T}}) \arrow[rr, "\beta"] \arrow[dr, "\alpha"] & & \mathrm{Ext}^0_{\mathcal{H}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}})}(T, T) \arrow[dl, "\gamma"] \\
& \mathrm{Ext}^0_{\mathcal{C}_{\mathbf{S}}}(T', T') &
\end{tikzcd}$$

The morphism β is an equivalence by the fully faithfulness of the functor $\phi\colon \mathcal{F}^{\mathrm{rel}} \to \mathcal{H}(Q, F, W)$. Using the geometric description of the endomorphisms of $\mathscr{G}_{\mathcal{T}}$ in [Chr21, Thm. 5.4,5.5] and of T' in Theorems 6.8 and 6.9, it is straightforward to check that α is also an equivalence. This implies that the map γ is also an equivalence.

Next, we show that τ is essentially surjective. Consider an indecomposable object $X \in \mathcal{C}_{\mathbf{S}}^{c}$. By Theorem 6.13, we have an equivalence $X \simeq M_\gamma$ for a matching curve γ in $\mathbf{S}$. The matching curve γ lifts uniquely to a pure matching datum $(\gamma, k[t_1])$ in $\mathbf{S}\backslash M$ in the sense of [Chr21, Def. 4.5], see [Chr21, Lem. 4.8]. The corresponding object $M_\gamma^{k[t_1]} \in \mathcal{D}(\mathscr{G}_{\mathcal{T}})$ lies in $\mathcal{F}^{\mathrm{rel}}$ and its image in $\mathcal{H}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}})$ is the desired lift of M_γ.

We have shown that $\mathrm{ho}\,\tau$ is an equivalence on extriangulated homotopy categories. The functor τ is thus also an exact functor between exact ∞-categories and further gives bijections between the subsets of morphisms given by inflations and deflations. To prove that τ is an equivalence of exact ∞-categories, it remains to show that τ is fully faithful, i.e. gives rise to equivalences between the mapping spaces, or equivalently, on all negative extensions groups. Consider two objects $X, X' \in \mathcal{F}^{\mathrm{rel}}$ with images $Y, Y' \in \mathcal{H}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}})$. By the equivalence of homotopy categories $\mathrm{ho}\,\mathcal{F}^{\mathrm{rel}} \simeq \mathrm{ho}\,\mathcal{C}_{\mathbf{S}}^{c}$, there exist equivalences $X \simeq M_\gamma^{k[t_1]}$ and $X' \simeq M_{\gamma'}^{k[t_1]}$ for two pure matching data $(\gamma, k[t_1]), (\gamma', k[t_1])$ in $\mathbf{S}\backslash M$. Note that $\tau(Y) \simeq M_\gamma$ and $\tau(Y') \simeq M_{\gamma'}$. The negative extension groups organize into the following commutative diagram,

$$\begin{tikzcd}
\mathrm{Ext}^{-i}_{\mathcal{D}(\mathscr{G}_{\mathcal{T}})}(X, X') \arrow[rr, "\simeq"] \arrow[dr, "\simeq"] & & \mathrm{Ext}^{-i}_{\mathcal{C}^{\mathrm{rel}}(Q_{\mathcal{T}}, F_{\mathcal{T}}, W_{\mathcal{T}})}(Y, Y') \arrow[dl] \\
& \mathrm{Ext}^{-i}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) &
\end{tikzcd}$$

where the left morphism is an isomorphism by the geometric description of the extension groups in Theorems 5.4 and 5.5 in [Chr21], using that $\mathrm{H}_i(k[t_1]) \simeq H_i(k[t_1^{\pm}]) \simeq k$ for all $i > 0$. The upper morphism is also an isomorphism, as shown in [Che23, Lem. 6.70]. This shows the desired fully faithfulness. $\square$

Remark 8.5. The conditions of Proposition 8.3 are satisfied for the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$, which can be seen as follows. The functor $\mathbf{Gi}_!$ is described geometrically in [Chr21, Prop 4.20]. A minor variation of that argument (where one glues the left adjoints instead of the right adjoints) yields a similar description of the functor $\mathbf{Gi}_*$. In the notation of loc. cit., we find for every edge e of the trivalent spanning graph $\mathcal{T}$ that $M_{c'_e}^{k[t_1]} \simeq \mathrm{ev}^!(k[t_1])$, with $\mathrm{ev}^!$ the right adjoint of ev_e (instead of the left adjoint), and c'_e the pure matching curve consisting of segments which each wrap exactly one step clockwise (instead of counterclockwise) around a vertex of $\mathcal{T}$. If e is an external edge, then c'_e is a boundary arc, and all boundary arcs arise this way. This relation between the left and right adjoints of $\mathbf{Gi}^*$ can also be explained by noting that $\mathbf{Gi}^*$ is a spherical functor whose cotwist functor permutes the components of $\mathcal{D}^{\mathrm{perf}}(\Pi_2(kF))$.

Lemma 8.6. *Let $\tau\colon (C, \mathbb{E}, \mathfrak{s}) \to (C', \mathbb{E}', \mathfrak{s}')$ be an extriangulated functor between extriangulated categories, such that C is Frobenius with finite dimensional Homs and extriangulated 2-Calabi–Yau. Assume further that*

1) *there are cluster tilting objects $T \in C$ and $T' \in C'$,*
2) *$\tau(T) = T'$,*
3) *$\tau\colon \mathrm{Hom}_C(T, T) \to \mathrm{Hom}_{C'}(T', T')$ is an isomorphism and*
4) *the functor τ is essentially surjective.*

Then τ is an extriangulated equivalence.

Remark 8.7. A version of Lemma 8.6 for triangulated categories (without the essential surjectivity assumption) was proven in section 4.5 of [KR08]

Proof of Lemma 8.6. Let $X \in C$ and $Y \in \mathrm{add}(T)$. By Lemma 5.16, there exist exact sequences $T_1 \to T_0 \to X$ and $X \to S_0 \to S_1$ in C with $T_0, T_1, S_0, S_1 \in \mathrm{add}(T)$. We get two morphisms between sequences of vector spaces which are exact by [GNP21, Thm. 3.5]:

$$\begin{tikzcd}
\mathrm{Hom}(Y,T_1) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(Y,T_0) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(Y,X) \arrow[r] \arrow[d] & \mathbb{E}(Y,T_1) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(Y,T_0) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\tau(Y),\tau(T_1)) \arrow[r] & \mathrm{Hom}(\tau(Y),\tau(T_0)) \arrow[r] & \mathrm{Hom}(\tau(Y),\tau(X)) \arrow[r] & \mathbb{E}(\tau(Y),\tau(T_1)) \arrow[r] & \mathbb{E}(\tau(Y),\tau(T_0))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(S_1,Y) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(S_0,Y) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(X,Y) \arrow[r] \arrow[d] & \mathbb{E}(S_1,Y) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(S_0,Y) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\tau(S_1),\tau(Y)) \arrow[r] & \mathrm{Hom}(\tau(S_0),\tau(Y)) \arrow[r] & \mathrm{Hom}(\tau(X),\tau(Y)) \arrow[r] & \mathbb{E}(\tau(S_1),\tau(Y)) \arrow[r] & \mathbb{E}(\tau(S_0),\tau(Y))
\end{tikzcd}$$

All vertical arrows, except the middle arrows, are equivalences by assumption 3) and the fact that T is rigid. By the five lemma, the middle vertical arrows above are also equivalences.

Similar arguments apply to the following two diagrams:

$$\begin{tikzcd}
\mathrm{Hom}(T_0,Y) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(T_1,Y) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(X,Y) \arrow[r] \arrow[d] & \mathbb{E}(T_0,Y) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(T_1,Y) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\tau(T_0),\tau(Y)) \arrow[r] & \mathrm{Hom}(\tau(T_1),\tau(Y)) \arrow[r] & \mathbb{E}(\tau(X),\tau(Y)) \arrow[r] & \mathbb{E}(\tau(T_0),\tau(Y)) \arrow[r] & \mathbb{E}(\tau(T_1),\tau(Y))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(Y,S_0) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(Y,S_1) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(Y,X) \arrow[r] \arrow[d] & \mathbb{E}(Y,S_0) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(Y,S_1) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\tau(Y),\tau(S_0)) \arrow[r] & \mathrm{Hom}(\tau(Y),\tau(S_1)) \arrow[r] & \mathbb{E}(\tau(Y),\tau(X)) \arrow[r] & \mathbb{E}(\tau(T),\tau(S_0)) \arrow[r] & \mathbb{E}(\tau(T),\tau(S_1))
\end{tikzcd}$$

We use the above constructed equivalences to show that τ is fully faithful on Homs and exact extensions. Let $X, X' \in C$. By Lemma 5.16, there exist exact sequences $T_1 \to T_0 \to X$ and $T'_1 \to T_0 \to X'$ in C with $T_0, T_1, T'_0, T'_1 \in \mathrm{add}(T)$. We form the following diagrams with horizontal exact sequences:

$$\begin{tikzcd}
\mathrm{Hom}(X',T_1) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(X',T_0) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(X',X) \arrow[r] \arrow[d] & \mathbb{E}(X',T_1) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(X',T_0) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\tau(X'),\tau(T_1)) \arrow[r] & \mathrm{Hom}(\tau(X'),\tau(T_0)) \arrow[r] & \mathrm{Hom}(\tau(X'),\tau(X)) \arrow[r] & \mathbb{E}(\tau(X'),\tau(T_1)) \arrow[r] & \mathbb{E}(\tau(X'),\tau(T_0))
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{Hom}(T_0', X) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(T_1', X) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(X', X) \arrow[r] \arrow[d] & \mathbb{E}(T_0', X) \arrow[r] \arrow[d, "\simeq"] & \mathbb{E}(T_1', X) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\tau(T_0'), \tau(X)) \arrow[r] & \mathrm{Hom}(\tau(T_1'), \tau(X)) \arrow[r] & \mathbb{E}(\tau(X'), \tau(X)) \arrow[r] & \mathbb{E}(\tau(T_0'), \tau(X)) \arrow[r] & \mathbb{E}(\tau(T_1'), \tau(X))
\end{tikzcd}
$$

The above shows that the non-central vertical morphisms are equivalences, and the five lemma yields that the central morphisms are also isomorphisms, showing the desired fully faithfulness of τ. $\square$

Example 8.8. Consider the A_2-quiver with a frozen vertex:

$$1 \xrightarrow{a} 2$$

The corresponding relative Ginzburg algebra $\mathscr{G}$ is the path algebra of the graded quiver

$$
\tilde{Q} \quad = \quad
\begin{tikzcd}
1 \arrow[loop left, "l"] \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "a^*"]
\end{tikzcd}
$$

with $|a| = 0$, $|a^*| = 1$, $|l| = 2$ and differential determined on the generators via $d(l) = a^*a$ and $d(a) = d(a^*) = 0$. We show in the following that the homology of $\mathscr{G}$ is generated by e_1, e_2, a, a^*, with e_i the lazy path at i, and thus finite dimensional. It follows that $\mathcal{D}^{\mathrm{perf}}(\mathscr{G}) = \mathcal{D}^{\mathrm{fin}}(\mathscr{G})$ and that the associated cosingularity category $\mathcal{D}^{\mathrm{perf}}(\mathscr{G})/\mathcal{D}^{\mathrm{fin}}(\mathscr{G})$ vanishes.

Let $X \in \mathscr{G}$ be a cycle, meaning that it satisfies $d(X) = 0$. We may split $X = \sum_{i,j=1}^{2} X_{i,j}$, with $X_{i,j}$ a cycle consisting of paths beginning at the vertex i and ending at the vertex j. We can write X as a k-linear sum of the generators, which are the paths of $\tilde{Q}$. We may thus write $X_{1,1}$ as $X_{1,1} = lY^1 + a^*aY^2 + \lambda e_1$, with $Y^1, Y^2 \in \mathscr{G}$ and $\lambda \in k$. Evaluating the condition $d(X_{1,1}) = 0$, we find $a^*aY^1 - a^*ad(Y^2) = 0$, hence $Y_1 = d(Y^2)$, and thus get that $X_{1,1} = d(lY^2) + \lambda e_1$ is homologous to λe_i. Similarly, we may write $X_{1,2} = aY$ for Y some cycle consisting of paths beginning and ending at 1. By the previous argument Y is homologous to e_1, and $X_{1,2}$ is hence homologous to a. Analogous arguments show that $X_{2,1}$ and $X_{2,2}$ are homologous to a^* or e_2.

8.2 The stable category of $\mathcal{C}_{\mathbf{S}}$ and the mapping class group action

Let $\mathbf{S}$ be a marked surface and $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ the associated (not Ind-complete) cosingularity category, considered as a Frobenius exact ∞-category. The associated stable category $\overline{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ is a stable ∞-category whose triangulated homotopy 1-category $\mathrm{ho}\,\overline{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ is equivalent to the stable category of the Frobenius extriangulated category $\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$. By Theorem 8.4, $\mathrm{ho}\,\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ is equivalent as an extriangulated category to the Higgs category. As shown in [Wu23, Cor. 5.19], the stable category $\mathrm{ho}\,\overline{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ is thus equivalent to the usual triangulated 2-Calabi–Yau cluster category associated with $\mathbf{S}$, considered for instance in [BZ11].

We further note that the mapping class group action on $\mathcal{C}_{\mathbf{S}}$ induces an action on the stable ∞-category $\overline{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$.

Proposition 8.9. *The mapping class group action on $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ from Theorem 4.15 induces an action on $\overline{\mathcal{C}}^{\mathrm{c}}_{\mathbf{S}}$ by automorphisms in* $\mathrm{ho}\,\mathrm{St}$, *and thus also an action on* $\mathrm{ho}\,\overline{\mathcal{C}}_{\mathbf{S}}$ *by equivalences of triangulated categories.*

Proof. The action of the mapping class group does not affect the values of the global sections at the external edges of the auxiliary trivalent spanning graph $\mathcal{T}$. It follows that the action preserves the set W from Proposition 5.5 and hence induces an action on the localization. $\square$

8.3 Generalized n-cluster categories of marked surfaces

The study of 2-Calabi-Yau cluster categories admits a generalization to triangulated n-Calabi-Yau categories with $n \geq 2$, called n-cluster categories. Their representation theoretic behavior is very similar to that of the 2-cluster categories. For example, there is a notion of n-cluster tilting objects which admit mutations similar to the 2-Calabi-Yau case. n-Cluster categories for $n \geq 2$ however do not categorify cluster algebras. They can be constructed in similar ways as the 2-cluster categories, for example as orbit categories or explicit geometric constructions. Amiot's construction of the generalized cluster category can also be used to construct n-Calabi–Yau categories with n-cluster tilting objects, see [Guo11]. A possible input for this is a higher Ginzburg algebra, meaning an $(n+1)$-Calabi-Yau version of the usual Ginzburg algebra.

We fix a marked surface $\mathbf{S}$ and choose an n-valent spanning graph $\mathcal{T}$ of $\mathbf{S}$. Its dual describes an n-angulation (decomposition into n-gons) of the surface. As shown in [Chr21], there is an associated relative higher Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ and a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$ with global sections its derived ∞-category $\mathcal{D}(\mathscr{G}_{\mathcal{T}})$. Its cosingularity category is given by the Verdier quotient

$$\mathcal{C}_{\mathbf{S}}^n \coloneqq \mathcal{D}(\mathscr{G}_{\mathcal{T}})/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})\,.$$

In the following, we survey how the results of this article generalize to $\mathcal{C}_{\mathbf{S}}^n$.

Locally at each vertex of $\mathcal{T}$, the spherical adjunction underlying the perverse schober $\mathcal{F}_{\mathcal{T}}$ is given by

$$f^*\colon \mathcal{D}(k) \longleftrightarrow \operatorname{Fun}(S^n, \mathcal{D}(k)) : f_*\,,$$

where f denotes the map $S^n \to *$. We now proceed in analogy with Section 4. Under the equivalence of ∞-categories $\operatorname{Fun}(S^n, \mathcal{D}(k)) \simeq \mathcal{D}(k[t_{n-1}])$, the functor f^* is identified with the pullback functor ϕ^* along $\phi\colon k[t_{n-1}] \xrightarrow{t_{n-1}\mapsto 0} k$. By Lemma 2.10, the quotient $\mathcal{D}(k[t_{n-1}])/\operatorname{Ind}\mathcal{D}(k[t_{n-1}])^{\mathrm{fin}}$ is equivalent to the derived ∞-category $\mathcal{D}(k[t_{n-1}^{\pm}])$ of $(n-1)$-periodic chain complexes.

Theorem 8.10. *There exist a vanishing-monadic and nearby-monadic $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}$ and a locally constant $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$ with generic stalk $\mathcal{D}(k[t_{n-1}^{\pm}])$ satisfying the following.*

i) There is a fiber and cofiber sequence of perverse schobers $\mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}} \to \mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}$.

ii) There exists an equivalence

$$\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{mnd}}) \simeq \operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})\,.$$

iii) There exists an equivalence

$$\mathcal{C}_{\mathbf{S}}^n \coloneqq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}) \simeq \mathcal{D}(\mathscr{G}_{\mathcal{T}})/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(\mathscr{G}_{\mathcal{T}})\,.$$

Proof. Analogous to the proofs of Proposition 4.5 and Theorem 4.8. □

We thus find the cosingularity category $\mathcal{C}_{\mathbf{S}}^n$ to be equivalent the topological Fukaya category of $\mathbf{S}$ with coefficients in the derived ∞-category of $(n-1)$-periodic chain complexes. In particular, the well-known 2-periodic topological Fukaya category of a surface is equivalent to $\mathcal{C}_{\mathbf{S}}^3$.

The ∞-category $\mathcal{C}_{\mathbf{S}}^n$ is smooth and proper as an $(n-1)$-periodic, or $2(n-1)$-periodic if n is even, stable ∞-category. The functor

$$\prod_{e\in\mathcal{T}_1^\partial} \mathrm{ev}_e\colon \mathcal{C}_{\mathbf{S}}^n = \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}) \longrightarrow \prod_{e\in\mathcal{T}_1^\partial} \mathcal{F}_{\mathcal{T}}^{\mathrm{clst}}(e) \simeq \prod_{e\in\mathcal{T}_1^\partial} \mathcal{D}(k[t_{n-1}^{\pm}])$$

further admits an n-Calabi-Yau structure, see [Chr26, Thm. 6.5].

A geometric model for $\mathcal{C}_{\mathbf{S}}^n$, describing (a priori a subset of) objects in $\mathcal{C}_{\mathbf{S}}^n$ in terms of matching data with local value $k[t_{n-1}^{\pm}][i]$, $0 \leq i \leq n-2$, is given in [Chr21]. The main difference to the geometric model for $\mathcal{C}^n_{\mathbf{S}}$ is thus that we may associate objects to matching curves equipped with a grading datum in $\mathbb{Z}/(n-1)\mathbb{Z}$. Note that [Chr21] only describes some of the morphism objects between the associated global sections (because the matching curves are not necessarily pure). One can prove analogs of Theorems 6.8 and 6.9 using a variation of the approach in [Chr21] applied directly to $\mathcal{C}^n_{\mathbf{S}}$. Furthermore, the proof of the geometrization Theorem 6.13 applies with minor modifications also to $\mathcal{C}^n_{\mathbf{S}}$, showing that all compact objects in $\mathcal{C}^n_{\mathbf{S}}$ arise from collections of graded matching curves in $\mathbf{S}\backslash M$.

As in Section 5.2, we can use the functor $\prod_{e\in\mathcal{T}_1^\partial}\mathrm{ev}_e$ to define an exact ∞-structure on $\mathcal{C}^{n,\mathrm{c}}_{\mathbf{S}}$ and hence also an extriangulated structure on the homotopy category $\mathrm{ho}\,\mathcal{C}^{n,\mathrm{c}}_{\mathbf{S}}$. This exact ∞-structure is again Frobenius.

References

[AL17] R. Anno and T. Logvinenko. Spherical DG-functors. *J. Eur. Math. Soc.*, 19:2577–2656, 2017.

[Ami09] C. Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. *Ann. Inst. Fourier (Grenoble)*, 59(6):2525–2590, 2009.

[AP21] C. Amiot and P. Plamondon. The cluster category of a surface with punctures via group actions. *Adv. Math.*, 389:107884, 2021.

[Bar15] C. Barwick. On exact ∞-categories and the theorem of the heart. *Compos. Math.*, 151(11):2160–2186, 2015.

[BCS20] T. Bozec, D. Calaque, and S. Scherotzke. Relative critical loci and quiver moduli. arXiv:2006.01069, 2020.

[BD19] C. Brav and T. Dyckerhoff. Relative Calabi-Yau structures. *Compos. Math.*, 155(2):372–412, 2019.

[BD21] C. Brav and T. Dyckerhoff. Relative Calabi–Yau structures II: shifted Lagrangians in the moduli of objects. *Sel. Math. New Ser.*, 27(63), 2021.

[BGT13] A. Blumberg, D. Gepner, and G. Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol. 17, 733-838*, 2013.

[BMR+06] A. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006.

[BS21] P. Balmer and G. Stevenson. Relative stable categories and birationality. *J. Lond. Math. Soc.*, 104(4):1765–1794, 2021.

[BZ11] T. Brüstle and J. Zhang. On the cluster category of a marked surface without punctures. *Algebra Number Theory*, 5:529–566, 2011.

[CDW23] M. Christ, T. Dyckerhoff, and T. Walde. Complexes of stable ∞-categories. arXiv:2301.02606, 2023.

[Che23] Xiaofa Chen. On exact dg categories. Ph.D. thesis, available at arXiv:2306.08231, 2023.

[Chr21] M. Christ. Geometric models for derived categories of Ginzburg algebras of n-angulated surfaces via local-to-global principles. arXiv:2107.10091*,* to appear in Selecta Math. New Ser., 2021.

[Chr22a] M. Christ. Ginzburg algebras of triangulated surfaces and perverse schobers. *Forum Math. Sigma*, 10(e8):72, 2022.

[Chr22b] M. Christ. Spherical monadic adjunctions of stable infinity categories. *Int. Math. Res. Not. IMRN*, 2023(15):13153–13213, 2022.

[Chr25a] M. Christ. Induction in perverse schobers and cluster tilting theory. arXiv:2509.01689, 2025.

[Chr25b] M. Christ. ∞-categorical group quotients via skew group algebras. arXiv:2501.13666, 2025.

[Chr26] M. Christ. Relative Calabi-Yau structures and perverse schobers on surfaces. *J. Noncommut. Geom.*, 20(3):871–951, 2026.

[Cis19] D. Cisinski. *Higher categories and homotopical algebra.* Cambridge Studies in Advanced Mathematics 180, Cambridge University Press, 2019.

[DK15] T. Dyckerhoff and M. Kapranov. Crossed simplicial groups and structured surfaces. *Stacks and categories in geometry, topology, and algebra, Contemp. Math., 643, Amer. Math. Soc., Providence, RI*, pages 37–110, 2015.

[DK18] T. Dyckerhoff and M. Kapranov. Triangulated surfaces in triangulated categories. *J. Eur. Math. Soc. (JEMS)*, 20(6):1473–1524, 2018.

[DKSS21] T. Dyckerhoff, M. Kapranov, V. Schechtman, and Y. Soibelman. Spherical adjunctions of stable ∞-categories and the relative S-construction. arXiv:2106.02873, 2021.

[Dyc21] T. Dyckerhoff. A categorified Dold-Kan correspondence. *Sel. Math. New Ser.*, 27(2), 2021.

[Efi17] A. Efimov. Categorical formal punctured neighborhood of infinity, I. arXiv:1711.00756, 2017.

[FG06] V. V. Fock and A. B. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Etudes Sci.*, 103(1):1–211, 2006.

[FST08] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. Part I: Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[FT18] S. Fomin and D. Thurston. Cluster algebras and triangulated surfaces. Part II: Lambda lengths. *Memoirs of Amer. Math. Soc.*, 225(1223), 2018.

[FWZ16] S. Fomin, L. Williams, and A. Zelevinsky. Introduction to cluster algebras. chapters 1-3. arXiv:1608.05735, 2016.

[FZ02] S. Fomin and A. Zelevinsky. Cluster algebras I: Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[GGV22] S. Ganatra, Y. Gao, and S. Venkatesh. Rabinowitz Fukaya categories and the categorical formal punctured neighborhood of infinity. arXiv:2212.14863, 2022.

[GNP21] M. Gorsky, H. Nakaoka, and Y. Palu. Positive and negative extensions in extriangulated categories. arXiv:2103.12482, 2021.

[Guo11] L. Guo. Cluster tilting objects in generalized higher cluster categories. *J. Pure Appl. Algebra*, 215(9):2055–2071, 2011.

[HKK17] F. Haiden, L. Katzarkov, and M. Kontsevich. Flat surfaces and stability structures. *Publ. Math. Inst. Hautes Etudes Sci,*, 126(1):247–318, 2017.

[IY08] O. Iyama and Y. Yoshino. Mutation in triangulated categories and rigid Cohen–Macaulay modules. *Invent. Math.*, 172:117–168, 2008.

[IY18] O. Iyama and D. Yang. Silting reduction and Calabi-Yau reduction of triangulated categories. *Trans. Am. Math. Soc.*, 370(11):7861–7898, 2018.

[Kel11] B. Keller. Deformed Calabi-Yau completions. *J. Reine Angew. Math*, 654:125–180, 2011.

[Kle22] J. Klemenc. The stable hull of an exact ∞-category. *Homology Homotopy Appl.*, 24(2):195–220, 2022.

[KN13] B. Keller and P. Nicolás. Weight structures and simple dg modules for positive dg algebras. *Int. Math. Res. Not.*, 2013(5):1028–1078, 2013.

[KR08] B. Keller and I. Reiten. Acyclic Calabi–Yau categories. *Compos. Math,*, 144(5):1332–1348, 2008.

[Kra15] H. Krause. Krull–Schmidt categories and projective covers. *Expositiones Mathematicae*, 33(4):535–549, 2015.

[KS14] M. Kapranov and V. Schechtman. Perverse Schobers. arXiv:1411.2772, 2014.

[KW23] B. Keller and Y. Wu. Relative cluster categories and Higgs categories with infinite-dimensional morphism spaces. arXiv:2307.12279, 2023.

[LF09] D. Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. *Proc. Lond. Math. Soc.*, 98(3):797–839, 2009.

[Lur] J. Lurie. Kerodon. available at https://kerodon.net/.

[Lur09] J. Lurie. *Higher Topos Theory.* Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ. MR 2522659, 2009.

[Lur17] J. Lurie. *Higher Algebra.* preprint, available on the author's webpage, 2017.

[Lur18] J. Lurie. *Spectral Algebraic Geometry.* preprint, available on the author's webpage, 2018.

[MSSS13] O. Mendoza, E. C. Sáenz, V. Santiago, and M. J. Souto Salori. Auslander-Buchweitz context and co-t-structures. *Appl. Categ. Struct.*, 21(5):417–440, 2013.

[Mul16] G. Muller. Skein and cluster algebras of marked surfaces. *Quantum Topol.*, 7:435–503, 2016.

[NP19] H. Nakaoka and Y. Palu. Extriangulated categories, Hovey twin cotorsion pairs and model structures. *Cah. Topol. Géom. Differ. Catég*, LX(2):117–193, 2019.

[NP20] H. Nakaoka and Y. Palu. External triangulation of the homotopy category of exact quasi-category. arXiv:2004.02479, 2020.

[OPS18] S. Opper, P. Plamondon, and S. Schroll. A geometric model of the derived category of a gentle algebra. arXiv:1801:09659, 2018.

[Pal08] Y. Palu. Cluster characters for 2-Calabi–Yau triangulated categories. *Ann. Inst. Fourier*, 58(6):2221–2248, 2008.

[QZ17] Y. Qiu and Y. Zhou. Cluster categories for marked surfaces: punctured case. *Compos. Math.*, 153(9):1779–1819, 2017.

[Smi15] I. Smith. Quiver Algebras as Fukaya Categories. *Geom. Topol.*, 19(5):2557–2617, 2015.

[Syl19] Z. Sylvan. Orlov and Viterbo functors in partially wrapped Fukaya categories. arXiv:1908.02317, 2019.

[Wu23] Y. Wu. Relative cluster categories and Higgs categories. *Adv. Math.*, 424:109040, 2023.

[Yeu16] W. Yeung. Relative Calabi-Yau completions. arXiv:1612.06352, 2016.

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