A gentle introduction to sheaves of stable ∞-categories

Lecture notes for a mincourse in Copenhagen, November 2025
(preliminary version)

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The goal of these lectures is to introduce a formalism for constructible sheaves of stable ∞-categories on graphs (spanning surfaces with boundary). As will see, this formalism can be used to approach representation theoretic questions. We will focus on a specific class of examples of such sheaves, whose stable ∞-categories of global sections describe derived categories of gentle algebras, or equivalently topological Fukaya categories. This class of examples is in some ways particularly simple but nevertheless exhibits many interesting phenomena.

In the first lecture, we recall the formalism for constructible sheaves in terms of functors out of the exit path category. The second lecture will introduce the specific class of examples giving rise to topological Fukaya categories. The third lecture will explain how widely known features of the representation theory of gentle algebras, the so-called geometric model, present themselves in terms of sheaf theory. In the last lecture, will study functors between different topological Fukaya categories and give an outlook on a broader classes of constructible sheaves called perverse schobers.

Preliminaries: We will assume basic knowledge of ∞-category theory and the language of stable ∞-categories. Prior exposure to the representation theory of gentle algebras will be helpful for lecture 3.

Contents

1 Lecture 1: Constructible sheaves on graphs

1.1 Marked surfaces and spanning graphs

Definition 1.1. A marked surface $(\mathbf{S}, M)$ consists of compact oriented topological surface $\mathbf{S}$ with non-empty boundary $\mathbf{S}$ and a finite set of marked points $M \subset \mathbf{S}$, such that every boundary component contains at least one marked point. We typically just write $\mathbf{S}$ for the marked surface $(\mathbf{S}, M)$.

By a graph $\mathbf{G}$, we will mean a graph with a finite set of vertices and edges. We allow external edges, meaning edges which are incident once to only a single vertex. Each internal edge of a ribbon graph consists of two halfedges, lying at the two vertices incident to the edge. For simplicity, we will not allow loops in graphs, meaning edges which are incident to the same vertex twice.

The geometric realization $|\mathbf{G}|$ of a graph $\mathbf{G}$ is the corresponding topological space, obtained by gluing together an interval for every edge along the vertices.

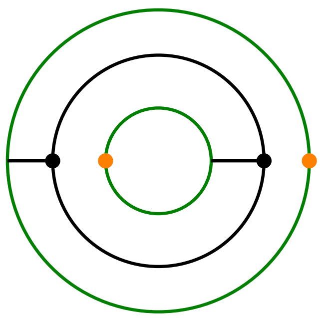
Definition 1.2. We call a graph $\mathbf{G}$ a spanning graph of a marked surface $\mathbf{S}$ if it is equipped with an embedding $i\colon |\mathbf{G}| \subset \mathbf{S}\backslash M$ satisfying that

- i is a homotopy equivalence,
- only the external endpoints of the external edges intersect the boundary $\partial\mathbf{S}$, and
- ι induces a bijection between the set of external edges of $\mathbf{G}$ and the connected components of $\partial\mathbf{S}\backslash M$.

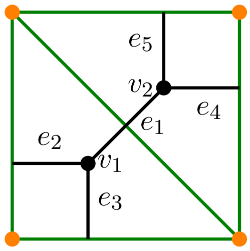
Remark 1.3. A ribbon graph is a graph $\mathbf{G}$ equipped with a cyclic order on the set of the incident half-edges at every vertex v of $\mathbf{G}$.

If $\mathbf{G}$ is a spanning graph of a marked surface $\mathbf{S}$, then it inherits a canonical ribbon graph structure, via the counterclockwise order induced by the orientation of $\mathbf{S}$.

Example 1.4. (1) The annulus (in green) with two marked points (in orange) together with a spanning graph (in black).



(2) A triangulation of a 4-gon and the dual trivalent spanning graph.



Note that every spanning graph is dual to a decomposition of the marked surface into polygons with corners at the marked points.

Definition 1.5. The exit path category $\mathrm{Exit}(\mathbf{G}) \in \mathrm{Cat}_\infty$ of a graph $\mathbf{G}$ is defined as the nerve of the 1-category with

- objects the vertices and edges of $\mathbf{G}$ and
- non-identity morphisms of the form $v \to e$ with v a vertex incident to an edge e. If e is a loop at v, then there are two morphisms $v \to e$ in $\mathrm{Exit}(\mathbf{G})$.

Example 1.6. The exit path category of the trivalent spanning graph from Example 1.4.(6) can be depicted as follows:

$$\begin{tikzcd}
e_2 & & & & e_5 \\
& v_1 \arrow[ul] \arrow[dl] \arrow[dr] & & v_2 \arrow[ur] \arrow[dl] \arrow[dr] & \\
e_3 & & e_1 & & e_4
\end{tikzcd}$$

1.2 Constructible sheaves on stratifies spaces

The above exit path category of a graph is a special case of a more general construction, taking as input a stratified space, see [Lur17, Section A.6].

A stratified space consists, roughly, of a topological space X together with nested subspaces[1] $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$, subject to conditions which depend on the context of interest (e.g. conical stratification, complex stratification, ...). We refer to the subspaces $X_i \backslash X_{i-1}$ as the strata of the stratification.

Example 1.7. Let $\mathbf{G}$ be a graph and denote by $\mathbf{G}_0$ its set of vertices. Then $\mathbf{G}_0 \subset |\mathbf{G}|$ is a stratified space. The 0-dim stratum is given by $\mathbf{G}_0$. The 1-dim stratum is given by the edges (without the vertices) in $|\mathbf{G}|$.

Given a stratified space X, the 0-simplicies of the simplicial set $\mathrm{Exit}(X)$ are the points of X and the 1-simplicies in X are paths in X which exit strata in the direction of lower indices. In the case $X = |\mathbf{G}|$, the resulting simplicial set is equivalent to $\mathrm{Exit}(\mathbf{G})$ from Definition 1.5 by using that each edge of $|\mathbf{G}|$ is contractible.

A sheaf on a stratified space X is called constructible if and only if its restriction to each stratum of X is a locally constant sheaf.

Theorem 1.8 ([Lur17] for $\mathcal{C} = \mathcal{S}$, [PT22])**.** *Let X be a sufficiently nice conically stratified space and $\mathcal{C}$ a compactly generated ∞-category. Then there exists an equivalence of ∞-categories*

$$\mathrm{Shv}_c(X, \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C})$$

[1] More generally, the subspaces can also be indexed by a poset.

between the ∞-category of $\mathcal{C}$-valued constructible sheaves on X and the ∞-category functors from the exit path ∞-category of X to $\mathcal{C}$.

In the case of constructible sheaves on graphs, we will leave the above identification between constructible sheaves and functors $\mathrm{Exit}(\mathbf{G}) \to \mathcal{C}$ implicit and simply refer to the latter as constructible sheaves. The value of such a functor $\mathcal{F}$ at an edge $e \in \mathrm{Exit}(\mathbf{G})$ describes the stalk of $\mathcal{F}$ at any point on the edge. Similarly, the value $\mathcal{F}(v)$ for $v \in \mathrm{Exit}(\mathbf{G})$ a vertex describes the stalk of $\mathcal{F}$ at v.

We will mostly be interested in the case that $\mathcal{C} = \mathrm{St}$, the compactly generated ∞-category of small stable ∞-categories and exact functors.

1.3 Global and lax sections

Definition 1.9. Let $\mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ be a constructible sheaf. The ∞-category of global sections $R\Gamma(\mathbf{G}, \mathcal{F})$ is defined as the limit $\lim \mathcal{F} \in \mathrm{St}$.

Recall that the forgetful functor $\mathrm{St} \to \mathrm{Cat}_\infty$ preserves limits.

Remark 1.10. Every functor $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{Cat}_\infty$ factors through the localization functor $N(\mathrm{Set}_\Delta) \to \mathrm{Cat}_\infty$. This means that $\mathcal{F}$ amounts to a strictly commuting diagram of simplicial sets. This follows from the observation that there are no non-degenerate n-simplicies for $n \geq 2$ in $\mathrm{Exit}(\mathbf{G})$.

We will always denote this factorization by $f\colon \mathrm{Exit}(\mathbf{G}) \to N(\mathrm{Set}_\Delta)$.

In Matteo's lecture, we saw the following:

Theorem 1.11 ([Lur, 03AB])**.** *Consider a functor $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \xrightarrow{f} N(\mathrm{Set}_\Delta) \to \mathrm{Cat}_\infty$. Then the limit of $\mathcal{F}$ is equivalent to the ∞-category of coCartesian sections of the coCartesian fibration $\Gamma(f) \to \mathrm{Exit}(\mathbf{G})$ called the Grothendieck construction (also called the weighted nerve in [Lur, 025X]).*

Definition 1.12. Consider a constructible sheaf $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$. The ∞-category of lax sections

$$\mathcal{L}(\mathbf{G}, \mathcal{F}) := \mathrm{Fun}(\mathrm{Exit}(\mathbf{G}), \Gamma(f)) \times_{\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}),\mathrm{Exit}(\mathbf{G}))} \{\mathrm{id}_{\mathrm{Exit}(\mathbf{G})}\}$$

is defined as the ∞-category of sections of the Grothendieck construction $p\colon \Gamma(f) \to \mathrm{Exit}(\mathbf{G})$.

Note that the inclusion of coCartesian sections into all sections defines (by definition) a fully faithful functor

$$R\Gamma(\mathbf{G}, \mathcal{F}) \hookrightarrow \mathcal{L}(\mathbf{G}, \mathcal{F})\,.$$

Remark 1.13. The ∞-category of lax sections of $\mathcal{F}$ describes the $(\infty, 2)$-categorical lax limit of $\mathcal{F}$, and thus has no analog in classical sheaf theory.

Example 1.14. Let

$$\mathbf{G}_1 = \ \cdot \!-\!\!-\!\!-\!\!-$$

be the ribbon graph with a single vertex v and a single edge e. Then $\mathrm{Exit}(\mathbf{G}_1) \simeq \Delta^1$. A functor $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}_1) \to \mathrm{St}$ thus amounts to an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between stable ∞-categories. Then $R\Gamma(\mathbf{G}_1, \mathcal{F}) \simeq \mathcal{F}(v) = \mathcal{A}$. An object in $\mathcal{L}(\mathbf{G}_1, \mathcal{F})$ however consists of a triple (X_a, X_b, η) with $X_a \in \mathcal{A} \subset \Gamma(f)$, $X_b \in \mathcal{B} \subset \Gamma(f)$ and $\eta\colon F(X_a) \to X_b$ a morphism in $\mathcal{B}$ (encoding a morphism $X_a \to X_b$ in $\Gamma(f)$). If the natural transformation η is a natural equivalence, then the lax section (X_a, X_b, η) is coCartesian and thus lies in the image of $\mathcal{A} \simeq R\Gamma(\mathbf{G}_1, \mathcal{F}) \hookrightarrow \mathcal{L}(\mathbf{G}_1, \mathcal{F})$.

Remark 1.15. Colimits in functor categories are computed pointwise, meaning that a diagram $Z^{\triangleright} \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is a colimit diagram if and only if its evaluation $Z^{\triangleright} \to \mathcal{D}$ at any $c \in \mathcal{C}$ is a colimit diagram. Similarly, (finite) colimits in the ∞-categories of sections $\mathcal{L}(\mathbf{G}, \mathcal{F})$ are computed pointwise in $\operatorname{Exit}(\mathbf{G})$. Using this fact, it is easy to see that $\mathcal{L}(\mathbf{G}, \mathcal{F})$ is a stable ∞-category.

We will next explain how local sections of $\mathcal{F}$, namely the stalks on $\mathbf{G}$, define lax sections of $\mathcal{F}$. For that, we will need to use relative Kan extensions.

1.4 Lax sections from relative Kan extension

We first sketch the definition of left Kan extensions. For simplicity, we only discuss the case of Kan extensions along inclusions, for a discussion of the general case, we refer to [Lur, Subsection 02Y7].

Let $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ be a fully faithful functor between ∞-categories and $F \colon \mathcal{C} \to \mathcal{D}$ a functor between ∞-categories. Let $X \in \mathcal{C}$ be an object. We denote $\mathcal{C}^0_{/X} = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/X} \subset \operatorname{Fun}(\Delta^1, \mathcal{C})$.

Definition 1.16. We say that F is the left Kan extension of $F|_{\mathcal{C}^0}$ if for each object $X \in \mathcal{C}$, the value $F(X)$ is given by the colimit of[2]

$$\mathcal{C}^0_{/X} \to \mathcal{C}^0 \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}\,.$$

The above definition is also known as the pointwise definition of Kan extensions: it describes how to compute the value of the Kan extension at every point $X \in \mathcal{C}$.

Example 1.17. Let

$$\mathcal{C}^0 = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \\ 2 & & \end{array} \qquad \hookrightarrow \qquad \mathcal{C} = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 2 & \longrightarrow & * \end{array}$$

Then the left Kan extension of $F \colon \mathcal{C}^0 \to \mathcal{D}$ evaluated at $* \in \mathcal{C}$ is given by the colimit (i.e. pushout) of F in $\mathcal{D}$.

Left Kan extensions form a functor that is the left adjoint to the pullback functor along $\mathcal{C}^0 \subset \mathcal{C}$:

Theorem 1.18 ([Lur, Cor. 030B])**.** *Suppose that $\mathcal{D}$ admits sufficient colimits. Then there exists an adjunction*

$$\operatorname{Lan} \colon \operatorname{Fun}(\mathcal{C}^0, \mathcal{D}) \longleftrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D}) : (\mathcal{C}^0 \subset \mathcal{C})^*$$

where Lan *maps each functor $\mathcal{C}^0 \to \mathcal{D}$ to its left Kan extension. Furthermore, the functor* Lan *is fully faithful (this fails if $\mathcal{C}^0 \to \mathcal{C}$ is not fully faithful).*

Suppose now that we are given functors $p \colon \mathcal{D} \to \mathcal{E}$ and $a \colon \mathcal{C} \to \mathcal{E}$, with $\mathcal{E}$ an ∞-category. If we are given a functor $F^0 \colon \mathcal{C}^0 \to \mathcal{D}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F^0"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r, "a"] & \mathcal{E} \end{tikzcd}$$

[2]More precisely, the canonical cone $\mathcal{C}^0_{/X} * \Delta^0 \to \mathcal{D}$ with tip $F(X)$ is a colimit cone.

there is a similar notion of p-relative left Kan extension, which is a lift of $\mathcal{C} \to \mathcal{E}$ along p of the following form:

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F^0"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r, "a"] \arrow[ur, "\mathrm{Lan}_p(F^0)"] & \mathcal{E} \end{tikzcd}$$

This works analogously to usual Kan extensions, these can again be characterized via a pointwise, now relative, colimit formula. The case of $\mathcal{E} = *$ corresponds to usual Kan extensions. The theory of relative colimits and Kan extensions is however too lengthy to unravel in this lecture. We invite the interested reader to consult [Lur, Subsection 02Z2].

Particularly relevant for us will be the case $\mathcal{C} = \mathcal{E}$ and $a = \mathrm{id}_{\mathcal{C}}$. We denote by

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C},\mathcal{C})} \{\mathrm{id}_{\mathcal{C}}\}$$

the ∞-category of sections of p and by

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}^0, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{C})} \{\mathcal{C}^0 \hookrightarrow C\}$$

the ∞-category of partial sections of p. Then p-relative left Kan extensions can be defined as follows:

Definition 1.19. Suppose that $\mathcal{D}$ admits sufficient colimits. Then the restriction functor

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}^0, \mathcal{D})$$

admits a left adjoint, called the p-relative left Kan extension functor Lan_p.

Example 1.20. Let $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ be a constructible sheaf. Let v be a vertex of $\mathbf{G}$. We fix $X \in \mathcal{F}(v)$. Consider the Grothendieck construction $p\colon \Gamma(f) \to \mathrm{Exit}(\mathbf{G})$. Note that $\mathcal{F}(v) \subset \Gamma(f)$ is the fiber of p at $v \in \mathrm{Exit}(\mathbf{G})$ and thus a full subcategory. We can thus consider X as a functor $\Delta^0 \to \Gamma(f)$. The left Kan extension $\mathrm{Lan}_p(X)$ defines a section of p:

$$\begin{tikzcd} \Delta^0 \arrow[r, "X"] \arrow[d, hook, "v"'] & \Gamma(f) \arrow[d, "p"] \\ \mathrm{Exit}(\mathbf{G}) \arrow[r, "\mathrm{id}"'] \arrow[ur, "\mathrm{Lan}_p(X)"] & \mathrm{Exit}(\mathbf{G}) \end{tikzcd}$$

Thus $\mathrm{Lan}_p(X) \in \mathcal{L}(\mathbf{G}, \mathcal{F})$ is a lax section. We can concretely describe $\mathrm{Lan}_p(X)$ as follows:

- We have $\mathrm{Lan}_p(X)(v) = X \in \mathcal{F}(v)$.
- For each vertex $v' \neq v$, we have $\mathrm{Lan}_p(X)(v') = 0 \in \mathcal{F}(v') \subset \Gamma(f)$. Roughly speaking, this is a consequence of the fact that there are no morphisms $v \to v'$ in $\mathrm{Exit}(\mathbf{G})$.
- Let $e_1, \dots, e_n$ be the edges incident to v. Then
$$\mathrm{Lan}_p(X)(e_i) = \mathcal{F}(v \to e_i)(X) \in \mathcal{F}(e_i)$$
and the morphism $\mathrm{Lan}_p(X)(v \to e_i)$ is coCartesian.
- For other edges $e' \neq e_1, \dots, e_n$, we again have $\mathrm{Lan}_p(X)(e') = 0$.

We can thus depict the lax section $\mathrm{Lan}_p(X)$ as follows:

$$\begin{tikzcd} 0 & 0 \arrow[l] \arrow[r] \arrow[d] & \mathcal{F}(v \to e_1)(X) & X \arrow[l, "!"'] \arrow[r, "!"] \arrow[d, "!"] & \mathcal{F}(v \to e_n)(X) & 0 \arrow[l] \arrow[r] \arrow[d] & 0 \\ & 0 & & \dots & & 0 & \end{tikzcd}$$

where the morphisms indicated by ! are coCartesian morphisms.

To summarize:

Lemma 1.21. *Let $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$.*

(1) Let $v \in \mathrm{Exit}(\mathbf{G})$ be a vertex. Then p-relative left Kan extension along $\Delta^0 \xrightarrow{v} \mathrm{Exit}(\mathbf{G})$ defines a fully faithful functor

$$\mathrm{Lan}_p(\text{-})\colon \mathcal{F}(v) \hookrightarrow \mathcal{L}(\mathbf{G}, \mathcal{F})\,,$$

embedding local sections at v into the ∞-category of lax sections. Further, Lan_p takes values in lax sections which are coCartesian locally at v.

(2) Let $e \in \mathrm{Exit}(\mathbf{G})$ be an edge. Then p-relative left Kan extension along $\Delta^0 \xrightarrow{e} \mathrm{Exit}(\mathbf{G})$ defines a fully faithful functor

$$\mathrm{Lan}_p(\text{-})\colon \mathcal{F}(e) \hookrightarrow \mathcal{L}(\mathbf{G}, \mathcal{F})\,.$$

For $X \in \mathcal{F}(e)$, the section $\mathrm{Lan}_p(X)$ vanishes at all $x \in \mathrm{Exit}(\mathbf{G})$, except at for $x = e$, where $\mathrm{Lan}_p(X)(e) \simeq X$.

The bigger stable ∞-category of lax sections of a constructible sheaf $\mathcal{F}$ can be helpful to study even when one is only interested in the ∞-category of global sections, since it forms the ambient category in which one can build global sections out of Kan extensions of local sections. We will see this in action in lecture 3.

2 Lecture 2: Topological Fukaya categories

In this lecture, we will introduce a specific class of constructible sheaves of stable ∞-categories on spanning ribbon graphs of surfaces. We will call them gentle sheaves. Their global sections are referred to as topological Fukaya categories. These categories were constructed in terms of a slightly different formalism (namely that of 2-Segal objects) by Dyckerhoff–Kapranov in [DK18, DK15]. The topological Fukaya categories are equivalent to the derived ∞-categories of graded gentle algebras [HKK17], as well as to the partially wrapped Fukaya categories of the marked surfaces considered as symplectic manifolds with stops at the marked points [LP20].

The form of gentle sheaves is very simple: they assign to each n-valent vertex the derived ∞-category of the A_{n-1}-quiver and to each edge the derived ∞-category $\mathcal{D}^{\mathrm{perf}}(k)$ of the base field k. What makes the construction non-trivial are the following two aspects:

- The A_{n-1}-quiver hides the rotational $\mathbb{Z}/n\mathbb{Z}$-symmetry of the n-gon. To get a uniquely defined ∞-category of global sections, we must show that all gentle sheaves on a given ribbon graph are equivalent, in particular explaining why they do not change under rotation of the n-gon. This is achieved by the Serre functor of $\mathcal{D}^{\mathrm{perf}}(A_{n-1})$, which acts as a kind of rotation, whose n-th power is the shift $[-2]$.

- We must show that the entire construction is independent on the choice of spanning graph. This is done by showing that contractions of ribbon graphs induce equivalences between the global sections.

2.1 Gentle sheaves on the n-spider

We fix a field k. We denote by $\mathcal{D}^{\mathrm{perf}}(A_n)$ the derived ∞-category of the abelian category of k-linear modules over the A_n-quiver. One can show that there exists an equivalence of stable ∞-categories $\mathcal{D}^{\mathrm{perf}}(A_n) \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$. An object in $\mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$ can be identified with a sequence of objects and morphisms $X_0 \to X_1 \to \cdots \to X_{n-1}$ in $\mathcal{D}^{\mathrm{perf}}(k)$.

Construction 2.1. As a warm-up, we turn the assignment

$$(X_0 \to X_1 \to \dots X_{n-1}) \mapsto \mathrm{fib}(X_i \to X_{i+1})$$

into an exact functor

$$\mathrm{fib}_{i,i+1} \colon \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k)) \to \mathcal{D}^{\mathrm{perf}}(k)\,.$$

For this, we compose the following two functors:

1) Consider the inclusion $\iota_{i,i+1} \colon \Delta^1 = \Delta^{\{i,i+1\}} \subset \Delta^{n-1}$ of the objects $i, i+1$. Then precomposition with $\iota_{i,i+1}$ defines a functor

$$\iota_{i,i+1}^* \colon \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k)) \to \mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k))\,.$$

This functors admits left and right adjoints, given by Kan extension, and is thus exact.

2) Let $\mathcal{C}$ be a stable ∞-category (for instance $\mathcal{C} = \mathcal{D}^{\mathrm{perf}}(k)$). Then the passage to the fiber defines an exact functor $\mathrm{fib} \colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$, see [Lur17, Rem. 1.1.1.7, 1.1.1.8].

Pulling back along the inclusions $\Delta^0 = \Delta^{\{0\}}, \Delta^0 = \Delta^{\{n-1\}} \hookrightarrow \Delta^{n-1}$, we similarly obtain exact functors $\pi_0, \pi_{n-1} \colon \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k)) \to \mathcal{D}^{\mathrm{perf}}(k)$ given on objects by the assignments

$$\pi_0 \colon (X_0 \to X_1 \to \dots X_{n-1}) \mapsto X_0$$

and

$$\pi_{n-1} \colon (X_0 \to X_1 \to \dots X_{n-1}) \mapsto X_{n-1}\,.$$

Definition 2.2. We denote by $\mathbf{G}_n$ the n-spider, given by the ribbon graph with a unique vertex v and n incident edges $e_1, \dots, e_n$. We define a constructible sheaf $\mathcal{F}_v \colon \mathrm{Exit}(\mathbf{G}_n) \to \mathrm{St}$ on $\mathbf{G}_n$ as follows:

- We set $\mathcal{F}_v(v) = \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$ and $\mathcal{F}_v(e_i) = \mathcal{D}^{\mathrm{perf}}(k)$ for all $1 \leq i \leq n$.
- We set

$$\mathcal{F}_v(v \to e_i) = \begin{cases} \pi_{n-1} & i = 1 \\ \mathrm{fib}_{n-i,n-i+1} & 2 \leq i \leq n-1 \\ \pi_0[1] & i = n\,. \end{cases}$$

We call $\mathcal{F}_v$ the gentle sheaf on the n-spider.

Remark 2.3. For $0 \leq i \leq n-2$, the functor $\mathrm{fib}_{i,i+1}$ is right adjoint to the functor

$$(\text{-}) \otimes_k S_i = (\text{-}) \otimes_k (\cdots \to 0 \to k \to 0 \to \dots) \colon \mathcal{D}^{\mathrm{perf}}(k) \to \mathcal{D}^{\mathrm{perf}}(A_n) \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))\,.$$

Here S_i corresponds to the simple module in the A_n-module category A_n arising from the the vertex $i+1$. The diagram $S_i = (\cdots \to 0 \to k \to 0 \to \dots) \in \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$ has the non-zero entry at the 0-simplex i.

Similarly, the functor π_{n-1} is right adjoint to

$$(\text{-}) \otimes S_{n-1} = (\text{-}) \otimes_k (\cdots \to 0 \to k) \colon \mathcal{D}^{\mathrm{perf}}(k) \to \mathcal{D}^{\mathrm{perf}}(A_n) \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$$

and $\pi_0[1]$ is right adjoint to

$$(\text{-}) \otimes_k S_n[-1] = (\text{-}) \otimes_k (k \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} k)[-1] \colon \mathcal{D}^{\mathrm{perf}}(k) \to \mathcal{D}^{\mathrm{perf}}(A_n) \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))\,.$$

Exercise 1. The left adjoint of the fiber functor $\mathrm{fib}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \to \mathcal{D}^{\mathrm{perf}}(k)$ can be shown to be given on objects by the assignment $X \mapsto (X \to 0)$. The left adjoint of restriction is given by left Kan extension. Using these two facts, prove the formula for the value of the left adjoint of $\mathrm{fib}_{i,i+1}$ at k given in Remark 2.3.

The Serre functor of $\mathcal{D}^{\mathrm{perf}}(A_n)$ cyclically permutes the objects $S_0, \dots, S_n$ up to shift. A discussion of Serre functors, especially in the ∞-categorical setting, would go beyond the scope of this lecture (see for instance [Chr23] for a definition of Serre functor). We next instead simply construct an autoequivalence U of $\mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$, and note that one can show this to be the negative suspension of the Serre functor.

Construction 2.4. Let $\mathcal{E}_1$ be the ∞-category of diagrams in $\mathcal{D}^{\mathrm{perf}}(k)$ of the form[3]

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
0 \arrow[r] & X_0 \arrow[r] & X_1 \arrow[r] & \dots \arrow[r] & X_{n-1}
\end{tikzcd}$$

and similarly $\mathcal{E}_2$ the ∞-category of diagrams in $\mathcal{D}^{\mathrm{perf}}(k)$ of the form

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \dots \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & X_0 \arrow[r] & X_1 \arrow[r] & \dots \arrow[r] & X_{n-1}
\end{tikzcd}$$

with all square being biCartesian.

There are apparent restriction functors $\mathcal{E}_2 \xrightarrow{r_2} \mathcal{E}_1 \xrightarrow{r_1} \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$. We observe that

- every diagram in $\mathcal{E}_1$ is the left Kan extension of its restriction to Δ^{n-1} along the inclusion $\Delta^{n-1} \subset \Delta^n \amalg_{\Delta\{0\}} \Delta^1$, and
- every diagram in $\mathcal{E}_2$ is the right Kan extension of its restriction to $\Delta^{n-1} \subset \Delta^n \amalg_{\Delta\{0\}} \Delta^1$ along the inclusion
$$\Delta^{n-1} \subset \Delta^n \amalg_{\Delta\{0\}} \Delta^1 \subset \Delta^n \times \Delta^1 \,.$$

Thus, the restriction functors r_1, r_2 are trivial fibrations, and in particular equivalences of ∞-categories.

We define an automorphism $U\colon \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k)) \to \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}^{\mathrm{perf}}(k))$ as the composite of $(r_2 \circ r_1)^{-1}$ with the restriction functor to $Y_0 \to \dots \to Y_{n-1}$.

It is straightforward to compute $U(S_i)$:

The diagram

$$\begin{tikzcd}
k[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & k \arrow[r] & k \arrow[r] & \dots \arrow[r] & k
\end{tikzcd}$$

shows that $U(S_n) \simeq S_0[-1]$.

The diagram

$$\begin{tikzcd}
k[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & k[-1] \arrow[r] \arrow[d] & \dots \arrow[r] & k[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & k
\end{tikzcd}$$

[3] Formally, this is a subcategory of the ∞-functor category $\mathrm{Fun}(\Delta^n \amalg_{\Delta\{n-1\}} \Delta^1, \mathcal{D}^{\mathrm{perf}}(k))$.

shows that $U(S_{n-1}) \simeq S_n[-1]$.

The diagram

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & k \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & k \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

shows that $U(S_i) \simeq S_{i+1}$ if $0 \leq i \leq n-2$.

Lemma 2.5. *For all $i \in \mathbb{Z}/n\mathbb{Z}$, there exists a natural equivalence*

$$\mathcal{F}_v(v \to e_i) \circ U \simeq \begin{cases} \mathcal{F}_v(v \to e_{i-1}) & 1 \leq i \leq n-1 \\ \mathcal{F}(v \to e_{n-1})[2] & i = n\,. \end{cases}$$

Thus, rotating the n-spider transforms the gentle sheaf $\mathcal{F}_v$ into an equivalent constructible sheaf.

Proof. Passing to left adjoints, we can equivalently show that

$$U^{-1} \circ \mathcal{F}(v \to e_i)^L \simeq \begin{cases} \mathcal{F}(v \to e_{i-1})^L & 1 \leq i \leq n-1 \\ \mathcal{F}(v \to e_{n-1})^L[-2] & i = n\,, \end{cases}$$

or that

$$\mathcal{F}(v \to e_i)^L \simeq \begin{cases} U \circ \mathcal{F}(v \to e_{i-1})^L & 1 \leq i \leq n-1 \\ U \circ \mathcal{F}(v \to e_{n-1})^L[-2] & i = n\,. \end{cases}$$

We show these equivalence in the case $n = 3$, the general case is analogous.

We have

$$\mathcal{F}(v \to e_1)^L = \pi_1^L \simeq (\text{-}) \otimes_k S_2$$

$$\mathcal{F}(v \to e_2)^L = \mathrm{fib}_{0,1}^L \simeq (\text{-}) \otimes_k S_1$$

$$\mathcal{F}(v \to e_3)^L = (\pi_0[1])^L \simeq (\text{-}) \otimes_k S_0[-1]$$

Thus

$$U \circ \mathcal{F}(v \to e_1)^L \simeq (\text{-}) \otimes_k U(S_1) \simeq (\text{-}) \otimes_k S_2[-1] \simeq \mathcal{F}(v \to e_3)^L$$
$$U \circ \mathcal{F}(v \to e_3)^L \simeq (\text{-}) \otimes_k U(S_2)[-1] \simeq (\text{-}) \otimes_k S_0[-2] \simeq \mathcal{F}(v \to e_2)^L[-2]$$

$$U \circ \mathcal{F}(v \to e_2)^L \simeq (\text{-}) \otimes_k U(S_0) \simeq (\text{-}) \otimes_k S_1 \simeq \mathcal{F}(v \to e_1)^L$$

as desired. □

2.2 Gentle sheaves and surface gradings

Let $\mathbf{S}$ be a marked surface and choose a spanning graph $\mathbf{G}$.

Definition 2.6. We call a constructible sheaf $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ a gentle sheaf if for each n-valent vertex v of $\mathbf{G}$ with corresponding fully faithful inclusion $\mathrm{Exit}(\mathbf{G}_n) \subset \mathrm{Exit}(\mathbf{G})$, the restriction $\mathcal{F}|_{\mathrm{Exit}(\mathbf{G}_n)}$ is equivalent in $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}_n), \mathrm{St})$ to the gentle sheaf $\mathcal{F}_v$ on the n-spider.

Example 2.7. Let $\mathbf{S}$ be the 4-gon. Let $\mathbf{G}$ be the trivalent dual ribbon graph of a triangulation of $\mathbf{S}$. Then a gentle sheaf on $\mathbf{G}$ is given by the following diagram:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{perf}}(k) & & & & & & \mathcal{D}^{\mathrm{perf}}(k) \\
& \mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[ul, "\mathrm{fib}"'] \arrow[dl, "\pi_0"] \arrow[drr, "\pi_1"] & & & & \mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[ur, "\mathrm{fib}"] \arrow[dll, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathcal{D}^{\mathrm{perf}}(k) & & & \mathcal{D}^{\mathrm{perf}}(k) & & & \mathcal{D}^{\mathrm{perf}}(k)
\end{tikzcd}$$

One can show that all gentle sheaves on this ribbon graph are equivalent.

Example 2.8. Let $\mathbf{G}$ be the spanning graph of the annulus from Example 1.4.(1). Then every gentle sheaf on $\mathbf{G}$ is equivalent to a diagram of the following form

$$\begin{tikzcd}
& & \mathcal{D}^{\mathrm{perf}}(k) & \\
\mathcal{D}^{\mathrm{perf}}(k) & \mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[l, "\pi_0"] \arrow[ur, "\mathrm{fib}"] \arrow[dr, "\pi_1"'] & \mathcal{D}^{\mathrm{perf}}(k) & \mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[l, "\pi_0"] \arrow[ul, "\mathrm{fib}"'] \arrow[dl, "{\pi_1[n]}"] \\
& & \mathcal{D}^{\mathrm{perf}}(k) &
\end{tikzcd}$$

for some $n \in \mathbb{Z}$. We note that these sheaves are however not equivalent to each other.

So far, we have not discussed gradings of surfaces (also known as line fields). A line field on a marked surface is a section of the projectivized tangent bundle of the surface (up to homotopy). The collection of line fields is a $H^1(\mathbf{S})$-torsor, they are determined by their winding numbers along loops in the surface. Each gentle sheaf determines a line field. For instance, the winding number of the line field on the annulus for the gentle sheaf from Example 2.8 along the clockwise loop is given by n. We next sketch how to determine the winding numbers of the line field from the gentle sheaf.

Interlude on transport and winding numbers

The winding numbers of the line field can be computed in terms of the so-called transport equivalences, which we introduced in [Chr23]. Consider an embedded loop $\gamma\colon S^1 = [0,1]/0 \sim 1 \to \mathbf{S}\backslash\mathbf{G}_0$. There is a decomposition of $\mathbf{S}$ into polygons, dual to the spanning graph $\mathbf{G}$. Intersecting γ with these polygons, we find that γ is the composite of a collection of curve segments embedded in these n-gons. The basepoint of γ is chosen arbitrarily to lie on an edge e of $\mathbf{G}$.

We first define the transport along these segments and compose these to obtain the transport $\mathcal{F}^{\to}(\gamma)$ of $\mathcal{F}$ along γ. Furthermore, each segment in a polygon is itself, up to homotopy fixing the endpoints, the composite of a collection of minimal segments that each go one step clockwise or counterclockwise. It thus suffices to define the transport for these elementary segments.

Consider an n-valent polygon with dual vertex v of $\mathbf{G}$ and incident edges $e_1, \dots, e_n$. If the elementary segment δ starts at the boundary component intersecting e_i and ends at the boundary components intersecting e_{i+1}, meaning it goes one step counterclockwise, we define $\mathcal{F}^{\to}(\delta)\colon \mathcal{F}(e_i) \to \mathcal{F}(e_{i+1})$ as the functor

$$\mathcal{F}(v \to e_{i+1}) \circ \mathcal{F}(v \to e_i)^L \,.$$

If δ instead goes clockwise, ending at the boundary component intersecting e_{i-1}, we set

$$\mathcal{F}^{\rightarrow}(\delta) = \mathcal{F}(v \to e_{i-1}) \circ \mathcal{F}(v \to e_i)^R .$$

Given a loop γ, the transport equivalence

$$\mathcal{D}^{\mathrm{perf}}(k) = \mathcal{F}(e) \xrightarrow{\mathcal{F}^{\rightarrow}(\gamma)} \mathcal{F}(e) = \mathcal{D}^{\mathrm{perf}}(k)$$

is necessarily given by a shift functor $[m]$ with $m \in \mathbb{Z}$, since these are the only k-linear autoequivalences of $\mathcal{D}^{\mathrm{perf}}(k)$. The number m describes the difference in the winding numbers along γ of the line field determined by $\mathcal{F}$ and the line field determined by the spanning graph $\mathbf{G}$ as in [Chr23, Ex. 4.8].

One can show that:

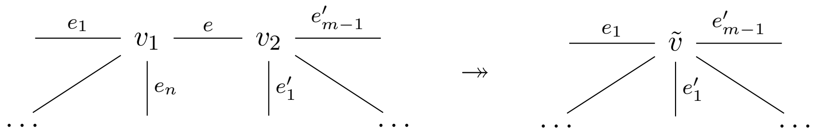
Proposition 2.9. *Let $\mathcal{F}, \mathcal{F}'\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ be two gentle sheaves. If for all loops γ in $\mathbf{S}\backslash\mathbf{G}_0$ there is a natural equivalence*

$$\mathcal{F}^{\rightarrow}(\gamma) \simeq (\mathcal{F}')^{\rightarrow}(\gamma)\,,$$

then the line fields on $\mathbf{S}$ induces by $\mathcal{F}, \mathcal{F}'$ coincide. This is the case if and only if there exists an equivalence $\mathcal{F} \simeq \mathcal{F}'$ in $\mathrm{Fun}(\mathrm{Exit}(\mathbf{G}), \mathrm{St})$.

2.3 Contractions of ribbon graphs

A contraction of an internal edge e of a ribbon graph is a local move of the following form:



Note that external edges are not allowed to be contracted (we will call the analogous process edge removal and discuss this later).

A contraction $c\colon \mathbf{G} \to \mathbf{G}'$ is a finite sequence of contractions of edges of $\mathbf{G}$ as above.

Construction 2.10. Let $c\colon \mathbf{G} \to \mathbf{G}'$ be a contraction between spanning graphs of a marked surface $\mathbf{S}$. Let $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ be a construcible sheaf on $\mathbf{G}$. We define a constructible sheaf $c_*(\mathcal{F})\colon \mathrm{Exit}(\mathbf{G}') \to \mathrm{St}$ on $\mathbf{G}'$ as follows:

We first suppose that c contracts a single edge e. We define $c_*(\mathcal{F})$ as follows:

- For v not incident to e, we set $c_*(\mathcal{F})(v) = \mathcal{F}(v)$. Similarly, for $e' \neq e$, wet set $c_*(\mathcal{F})(e') = \mathcal{F}(e')$ and for $v \to e'$ a morphism in $\mathrm{Exit}(\mathbf{G})$, we set $c_*(\mathcal{F})(v \to e') = \mathcal{F}(v \to e')$.
- Let v_1, v_2 be the two vertices incident to e. These are contracted to a unique vertex $\tilde{v}$ of $\mathbf{G}'$. We define $c_*(\mathcal{F})(\tilde{v})$ as the pullback

$$\begin{tikzcd} c_*(\mathcal{F})(\tilde{v}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F}(v_2) \arrow[d, "\mathcal{F}(v_2 \to e)"] \\ \mathcal{F}(v_1) \arrow[r, "\mathcal{F}(v_1 \to e)"] & \mathcal{F}(e) \end{tikzcd} \tag{1}$$

- For any edge e' incident to $\tilde{v}$ in $\mathbf{G}'$, we obtain $c_*(\mathcal{F})(\tilde{v} \to e')$ by composing functors $c_*(\mathcal{F})(\tilde{v}) \to \mathcal{F}(v_i)$ and $\mathcal{F}(v_i) \to \mathcal{F}(e') = c_*(\mathcal{F})(e')$.

In the case that c contracts multiple edges, we repeatedly apply the above construction on any order of the contracted edges.

Lemma 2.11. *Let $c\colon \mathbf{G} \to \mathbf{G}'$ be a contraction of ribbon graphs and $\mathcal{F}\colon \operatorname{Exit}(\mathbf{G}) \to \mathrm{St}$. There exists an equivalence of ∞-categories of global sections*

$$R\Gamma(\mathbf{G}, \mathcal{F}) \simeq R\Gamma(\mathbf{G}', c_*(\mathcal{F}))\,.$$

Proof. The contraction c induces a functor $\operatorname{Exit}(c)\colon \operatorname{Exit}(\mathbf{G}) \to \operatorname{Exit}(\mathbf{G}')$, mapping each vertex to the vertex it gets contracted into. We observe that $c_*(\mathcal{F})$ is the right Kan extension of $\mathcal{F}$ along $\operatorname{Exit}(c)$.

For instance, in the case of a contraction of a single edge e with two incident vertices v_1, v_2, we have that v_1, v_2, e are all mapped to $\tilde{v}$ and thus the inclusion of the cospan $v_1 \to e \leftarrow v_2$ into $\operatorname{Exit}(\mathbf{G})_{\tilde{v}/}$ is cofinal. Thus the right Kan extension evaluated at $\tilde{v}$ is computed exactly as the pullback (1).

The passage to the limit $R\Gamma(\mathbf{G}', c_*(\mathcal{F})) = \lim c_*(\mathcal{F})$ is right Kan extension along $\operatorname{Exit}(\mathbf{G}') \to *$. The desired equivalence thus follows from the fact that the composite of two right Kan extensions is again a right Kan extension. $\square$

Lemma 2.12. *Let $c\colon \mathbf{G} \to \mathbf{G}'$ be a contraction of ribbon graphs. If $\mathcal{F}$ is a gentle sheaf on $\mathbf{G}$, then $c_*(\mathcal{F})$ is a gentle sheaf on $\mathbf{G}'$.*

Proof sketch. Suppose that c contracts the edge e incident to two vertices v_1, v_2 of valency n and m. The contracted vertex $\tilde{v}$ has valency $n + m - 2$. Applying the limit preserving functor $\operatorname{Fun}(\text{-}, \mathcal{D}^{\mathrm{perf}}(k))\colon (\mathrm{Cat}^{\infty})^{\mathrm{op}} \to \mathrm{St}$ to the pushout square

$$\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Delta^{n-2} \arrow[d] \\
\Delta^{m-2} \arrow[r] & \Delta^{n+m-4}
\end{tikzcd}$$

we obtain a pullback square

$$\begin{tikzcd}
\operatorname{Fun}(\Delta^{n+m-4}, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Fun}(\Delta^{n-2}, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[d] \\
\operatorname{Fun}(\Delta^{m-2}, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[r] & \operatorname{Fun}(\Delta^{0}, \mathcal{D}^{\mathrm{perf}}(k))
\end{tikzcd}$$

which implies $c_*(\mathcal{F})(\tilde{v}) \simeq \mathcal{F}(v_1) \times_{\mathcal{F}(e)} \mathcal{F}(v_2) \simeq \operatorname{Fun}(\Delta^{n+m-4}, \mathcal{D}^{\mathrm{perf}}(k))$. We leave to the reader the computation of the functors in $c_*(\mathcal{F})$, showing that $c_*(\mathcal{F})$ indeed is gentle. $\square$

Using this, one shows:

Theorem 2.13. *Let $\mathbf{S}$ be a marked surface. Choosing two different spanning graphs $\mathbf{G}, \mathbf{G}'$ of $\mathbf{S}$ and two gentle sheaves $\mathcal{F}\colon \operatorname{Exit}(\mathbf{G}) \to \mathrm{St}$ and $\mathcal{F}'\colon \operatorname{Exit}(\mathbf{G}') \to \mathrm{St}$ there exists an equivalence of ∞-categories of global sections*

$$R\Gamma(\mathbf{G}, \mathcal{F}) \simeq R\Gamma(\mathbf{G}', \mathcal{F}')$$

if and only if the corresponding line fields agree.

We call the equivalence class of the ∞-category $R\Gamma(\mathbf{G}, \mathcal{F})$ the $\mathcal{D}^{\mathrm{perf}}(k)$-valued topological Fukaya category of $\mathbf{S}$ (considered as equipped with the corresponding line field).

Proof. This follows from Lemma 2.11 and the fact that any two spanning graphs are connected via a zig-zag of contractions and Proposition 2.9. $\square$

Remark 2.14. One can refine Theorem 2.13 to a construction of the topological Fukaya category whose result is unique up to contractible choice, see [DK15].

Remark 2.15. The above discussion readily translates when replacing the coefficients $\mathcal{D}^{\mathrm{perf}}(k)$ by any other stable ∞-category $\mathcal{D}$, such as the stable ∞-category of spectra Sp.

Remark 2.16. Suppose that the stable ∞-category of coefficients $\mathcal{D}$ is 2-periodic (and thus $\mathrm{id}_{\mathcal{D}} \simeq [2]$). Then the winding numbers of the gentle sheaf only matter up to parity. In this case, there is a distinguished topological Fukaya category, which is associated with any line field all of whose winding numbers are even. These 2-periodic topological Fukaya categories were constructed in [DK18] in the k-linear setting and in [Lur15] for 2-periodic stable ∞-categories.

Choosing $\mathcal{D} = \mathcal{D}^{\mathrm{perf}}(k[t_1^{\pm}]) \simeq \mathcal{D}^{\mathrm{perf}}(k)/[1]$ the 1-periodic derived ∞-category (which is a 2-periodic ∞-category), the $\mathcal{D}$-valued topological Fukaya category describes the Higgs category categorifying the cluster algebra of the corresponding marked surface with coefficients in the boundary arcs, see [Chr22]

3 Lecture 3: The geometric model via gluing

Many (if not all) aspects of the representation theory of derived categories of (graded) gentle algebras can be described in terms of the combinatorial geometry of the corresponding marked surface. For instance, the indecomposable objects can be classified in terms of certain homotopy classes of curves (with decorations) in the marked surface. Such descriptions are referred to as a geometric model [OPS18].

In this lecture, we will explain how this geometric model can be seen to arise from sheaf theory. We will begin by matching the gluing of compatible local sections to a global section on the geometric side with the composition of local curve segments to a global curve. As a side note, we emphasize that this is the point where it becomes important that we formulated the construction in terms of sheaves (whose local sections can be readily glued), as opposed to cosheaves. The cosheaf language is typically the default in the literature, see for instance [HKK17, GPS24], and has other advantages.

3.1 Objects from curves

For simplicity, we will only associate objects with embedded curves. We start with open embedded curves, called arcs.

Definition 3.1. Let $\mathbf{S}$ be a marked surface. An arc in $\mathbf{S}$ consists of an embedded curve $\gamma\colon [0,1] \to \mathbf{S}$ satisfying that:

- The endpoints of γ lie in $\partial\mathbf{S}\backslash M$.
- The curve γ does not intersect $\partial\mathbf{S}$ except at the endpoints.
- The curve γ does not cut out a monogon (or equivalently γ is not contractible to a point in $\partial\mathbf{S}$).

We consider arcs up to homotopies relative $\partial\mathbf{S}\backslash M$[4].

Definition 3.2. Let $\mathbf{S}$ be a marked surface equipped with an ideal triangulation. An arc segment in $\mathbf{S}$ consists of an arc embedded in one of the triangles of $\mathbf{S}$, considered itself as a marked surface.

Definition 3.3. An ideal triangulation of a marked surface consists of a maximal collection of non-intersecting[5] arcs.

[4] This means that the endpoints are moved by the homotopies at most on the same component of $\partial\mathbf{S}\backslash M$.

[5] Meaning they have representatives that do not intersect.

We note that the set of ideal triangulations of a marked surface is canonically in bijection with the set of trivalent spanning graphs up to homotopy.

Remark 3.4. Each arc γ can be uniquely written as the composite of a finite collection of arc segments $\delta_1, \dots, \delta_n$. The arc segments are obtained by intersecting a representative of γ with the ideal triangles of the triangulation.

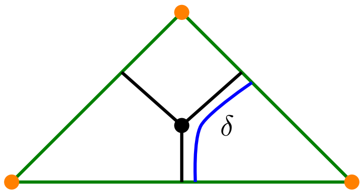


Figure 1: A triangle dual to a trivalent vertex of a ribbon graph. There are exactly three arc segments in the triangle, one connecting each pair of edges of the triangle. One of the three arc segments, denoted δ, is depicted here.

We now fix a marked surface $\mathbf{S}$ with a trivalent spanning graph $\mathbf{G}$, dual to an ideal triangulation. Let $\mathcal{F}\colon \operatorname{Exit}(\mathbf{G}) \to \operatorname{St}$ be a gentle sheaf.

Lax sections from arc segments

Fix a triangle of $\mathbf{S}$, containing the trivalent vertex v of $\mathbf{G}$. Then $\mathcal{F}(v) = \operatorname{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \simeq \mathcal{D}^{\mathrm{perf}}(A_2)$. Up to shifts, $\mathcal{F}(v)$ thus contains the following three indecomposable modules:

$$0 \to k, \quad k \xrightarrow{\mathrm{id}} k, \quad k \to 0\,.$$

We will match these objects with the three arc segments of the triangle.

Let e_1, e_2, e_3 be the three edges incident to v. At v, e_1, e_2, e_3, up to shift, $\mathcal{F}$ is given by $\mathcal{F}_v$, see above, with

$$\mathcal{F}_v(v \to e_i) = \begin{cases} \pi_1 & i = 1 \\ \mathrm{fib}_{0,1} & i = 2 \\ \pi_0[1] & i = 3 \end{cases}$$

Thus

$$\mathcal{F}(v \to e_i)(0 \to k) = \begin{cases} k & i = 1 \\ k[-1] & i = 2 \\ 0 & i = 3 \end{cases}$$

$$\mathcal{F}(v \to e_i)(k \to 0) = \begin{cases} 0 & i = 1 \\ k & i = 2 \\ k[1] & i = 3 \end{cases}$$

$$\mathcal{F}(v \to e_i)(k \to k)[-1] = \begin{cases} k[-1] & i = 1 \\ 0 & i = 2 \\ k & i = 3 \end{cases}$$

Note that each indecomposable object evaluates non-trivially at exactly two of the three edges incident to v.

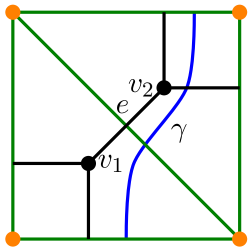


Figure 2: An arc in the 4-gon.

Definition 3.5. Let δ be a segment in the ideal triangle at v, starting at an edge e_i and ending at an edge e_{i+1} for some $i \in \mathbb{Z}/3\mathbb{Z}$. We define $L_\delta^v \in \mathcal{F}(v)$ as follows:

$$L_\delta^v = \begin{cases} 0 \to k & i = 1 \\ k \to 0 & i = 2 \\ (k \to k)[-1] & i = 3\,. \end{cases}$$

Let $p\colon \Gamma(f) \to \mathrm{Exit}(\mathbf{G})$ denote the Grothendieck construction of $\mathcal{F}$. We turn L_δ^v into a lax section of $\mathcal{F}$ via p-relative left Kan extension:

Definition 3.6. We define $L_\delta = \mathrm{Lan}_p(L_\delta^v) \in \mathcal{L}(\mathbf{G}, \mathcal{F})$ as the p-relative left Kan extension of $L_\delta^v \in \mathcal{F}(v) \subset \Gamma(\mathcal{F})$ along the inclusion

$$\Delta^0 \xrightarrow{v} \mathrm{Exit}(\mathbf{G})\,.$$

As we noted in lecture 1, we have $L_\delta(v) \simeq L_\delta^v \in \mathcal{F}(v)$ and $L_\delta(e_i) \simeq \mathcal{F}(v \to e_i)(L_\delta^v)$. Further, for $v' \neq v$ and $e \neq e_1, e_2, e_3$, we have $L_\delta(v') \simeq 0$ and $L_\delta(e) \simeq 0$.

Gluing lax sections along arcs

We first spell out the construction in the case that $\mathbf{S}$ is the 4-gon, with two ideal triangles. We choose $\mathbf{G}$ and the arc γ (in blue) as in Figure 2

Let $\mathcal{F}$ be the gentle sheaf from Example 2.7:

$$\mathcal{F} = \begin{tikzcd} \mathcal{D}^{\mathrm{perf}}(k) & & & & & \mathcal{D}^{\mathrm{perf}}(k) \\ & \mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[ul, "\mathrm{fib}"] \arrow[dl, "\pi_0"] \arrow[dr, "\pi_1"] & & & \mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k)) \arrow[ur, "\mathrm{fib}"] \arrow[dll, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ \mathcal{D}^{\mathrm{perf}}(k) & & \mathcal{D}^{\mathrm{perf}}(k) & & & \mathcal{D}^{\mathrm{perf}}(k) \end{tikzcd}$$

Let δ_1, δ_2 be the two arcs segments of γ. Then $L_{\delta_1}^{v_1} = (k \to k) \in \mathcal{F}(v_1)$ and $L_{\delta_2}^{v_2} = (k \to 0) \in \mathcal{F}(v_2)$. Thus

$$L_{\delta_1}(e) \simeq \pi_1(k \to k) \simeq k \simeq \pi_0(k \to 0) \simeq L_{\delta_2}(e)\,.$$

Let L_e be given by the p-relative left Kan extension of $k \in \mathcal{D}^{\mathrm{perf}}(k) = \mathcal{F}(e) \subset \Gamma(\mathcal{F})$ along the inclusion

$$\Delta^0 \xrightarrow{e} \mathrm{Exit}(\mathbf{G})\,.$$

The lax section L_e vanishes everywhere except at e, meaning that $L_e(e) \simeq k$ and $L_e(x) \simeq 0$ for all $e \neq x \in \mathrm{Exit}(\mathbf{G})$. There are inclusions of lax sections $L_e \hookrightarrow L_{\delta_1}$ and $L_e \hookrightarrow L_{\delta_2}$, which evaluate at e to the equivalences $L_{\delta_1}(e), L_{\delta_2}(e) \simeq k \simeq L_e(e)$. We can depict the above lax sections as the following diagrams in $\Gamma(\mathcal{F})$:

$$
\begin{tikzcd}
 & 0 & & & & 0 \\
L_e = & & 0 \arrow[ul] \arrow[dl] \arrow[dr] & & 0 \arrow[ur] \arrow[dl] \arrow[dr] & \\
 & 0 & & k & & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 & & & & 0 \\
L_{\delta_1} = & & (k \to k) \arrow[ul, "!"'] \arrow[dl, "!"'] \arrow[dr, "!"] & & 0 \arrow[ur] \arrow[dl] \arrow[dr] & \\
 & k & & k & & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 & & & & k \\
L_{\delta_2} & & 0 \arrow[ul] \arrow[dl] \arrow[dr] & & (k \to 0) \arrow[ur, "!"] \arrow[dl, "!"'] \arrow[dr, "!"] & \\
 & 0 & & k & & 0
\end{tikzcd}
$$

We now define $X_\gamma \in \mathcal{L}(\mathbf{G}, \mathcal{F})$ as the pushout of the following diagram:

$$
\begin{tikzcd}
 & L_e \arrow[dl, hook] \arrow[dr, hook] & \\
L_{\delta_1} & & L_{\delta_2}
\end{tikzcd}
$$

Limits in $\mathcal{L}(\mathbf{G}, \mathcal{F})$ are computed pointwise, meaning that $X_\gamma(y)$ is the pullback $L_{\delta_1}(y) \times_{L_e(y)} L_{\delta_2}(y)$ for all $y \in \mathrm{Exit}(\mathbf{G})$. We can thus depict X_γ as follows:

$$
\begin{tikzcd}
0 & & & & k \\
 & (k \to k) \arrow[ul, "!"'] \arrow[dl, "!"'] \arrow[dr] & & (k \to 0) \arrow[ur, "!"] \arrow[dl, "!"'] \arrow[dr, "!"] & \\
k & & k & & 0
\end{tikzcd}
$$

We observe that the section X_γ is coCartesian. This is no coincidence, X_γ was glued from sections supported at different vertices which are coCartesian at their respective vertices. Thus $X_\gamma \in R\Gamma(\mathbf{G}, \mathcal{F}) \subset \mathcal{L}(\mathbf{G}, \mathcal{F})$ defines the desired object of the topological Fukaya category associated with the arc γ.

The above construction readily generalizes to arbitrary arcs in arbitrary marked surfaces equipped with a trivalent spanning graph (one can of course equally treat non-trivalent spanning graphs): one constructs iterated pushouts, gluing in the local section corresponding to one arc segment after the next. The result is a global section X_γ associated with every arc γ. In any systematic development of this story, one must further equip the arcs with gradings relative to the line field (which form a $\mathbb{Z}$-torsor).

Remark 3.7. We observe that the support of a global section associated with an arc, meaning the objects of $\mathrm{Exit}(\mathbf{G})$ where the section evaluates non-trivially, is described exactly by the arc.

Exercise 2. Describe an analog of the above construction for closed embedded curves γ along which the transport $\mathcal{F}^{\rightarrow}(\gamma)$ equivalent to the identity. Explain why one can associate an object with a closed embedded curve equipped with a rank n local system on the curve (meaning an automorphisms of $k^{\oplus n} \in \mathcal{D}^{\mathrm{perf}}(k)$).

3.2 Morphisms from intersections

Given a stable ∞-category $\mathcal{C}$, there is a spectrally enriched 'derived Hom', meaning a functor

$$\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$$

valued in the stable ∞-category of spectra. This functor is exact in each component. For two objects $X, Y \in \mathcal{C}$, we have

$$\pi_i \operatorname{Mor}_{\mathcal{C}}(X,Y) \simeq \operatorname{Ext}^{-i}_{\mathrm{h}\,\mathcal{C}}(X,Y)$$

for all $i \in \mathbb{Z}$.

Given two arcs γ, γ' in a marked surface, the corresponding global sections $X_\gamma, X_{\gamma'}$ arise as finite limits of diagrams built out of the lax sections associated with the segments of γ, γ'. Using that $\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-})$ is exact in each component, we can thus compute the extension groups between the global sections out of the derived Homs between the lax sections associated with segments.

The derived Homs between the lax sections of segments can be computed locally, i.e. in $\mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k))$, using the following Lemma:

Lemma 3.8. *Let δ be a segment lying at a vertex v and $L' \in \mathcal{L}(\mathbf{G},\mathcal{F})$. Then*

$$\mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L_\delta, L') \simeq \mathrm{Mor}_{\mathcal{F}(v)}(L_\delta(v), L'(v))\,.$$

In particular, if $L' = L_{\delta'}$ for a segment δ lying a vertex $v' \neq v$, then

$$\mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L_\delta, L_{\delta'}) \simeq 0\,.$$

Proof. Recall that L_δ was defined as the p-relative left Kan extensions of $L_\delta(v)$. The desired equivalence follows from the fact that p-relative left Kan extension is left adjoint to restriction of sections to v. □

We have

$$\mathrm{Mor}_{\mathrm{Fun}(\Delta^1,\mathcal{D}^{\mathrm{perf}}(k))}(0\to k, k\to 0) \simeq 0\,, \quad \mathrm{Mor}_{\mathrm{Fun}(\Delta^1,\mathcal{D}^{\mathrm{perf}}(k))}(k\to 0, 0\to k) \simeq k[1]$$

and

$$\mathrm{Mor}_{\mathrm{Fun}(\Delta^1,\mathcal{D}^{\mathrm{perf}}(k))}(0\to k, 0\to k) \simeq k\,.$$

Using the rotation symmetry, i.e. the Serre functor, this tells us all derived Homs between the indecomposable objects in $\mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k))$. They count boundary intersections of the corresponding segments in the triangle:

Definition 3.9. Let $\gamma \neq \gamma'$ be two arcs in a marked surface $\mathbf{S}$. We choose representatives of γ, γ' with the minimal number of intersections.

(1) A directed boundary intersection from γ to γ' is an intersection of both γ and γ' with a boundary component B of $\mathbf{S}\backslash M$, such that the intersection of γ with B precedes the intersection of γ' with B in the clockwise direction.

Figure 3: On the left: a crossing of two arcs γ, γ'. On the right: a directed boundary intersection from γ to γ'.

(2) A crossing of γ, γ' consists of an intersection of γ, γ' in the interior of the surface.

Example 3.10. Let γ be as in Figure 2 and γ' the unique arc with a crossing with γ (going from the left vertical boundary edge to the right vertical boundary edge). We let $\mathcal{F}$ be as in Example 2.7. We can depict $X_{\gamma'}$ as follows:

$$\begin{tikzcd}
k & & & & 0 \\
& (0 \to k[1]) \arrow[ul, "!"'] \arrow[dl, "!"] \arrow[dr] & & (k[1] \to k[1]) \arrow[ur, "!"] \arrow[dl, "!"'] \arrow[dr, "!"] & \\
0 & & k[1] & & k[1]
\end{tikzcd}$$

Using that $X_\gamma \simeq L_{\delta_1} \times_{L_e} L_{\delta_2}$, we find a pushout diagram:

$$\begin{tikzcd}
\mathrm{Mor}(L_e, X_{\gamma'}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Mor}(L_{\delta_1}, X_{\gamma'}) \arrow[d] \\
\mathrm{Mor}(L_{\delta_2}, X_{\gamma'}) \arrow[r] & \mathrm{Mor}(X_\gamma, X_{\gamma'})
\end{tikzcd}$$

We further have

$$\mathrm{Mor}(L_e, X_{\gamma'}) \simeq \mathrm{Mor}_{\mathcal{D}^{\mathrm{perf}}(k)}(k, X_{\gamma'}(e)) \simeq \mathrm{Mor}_{\mathcal{D}^{\mathrm{perf}}(k)}(k, k[1]) \simeq k[1]$$

$$\mathrm{Mor}(L_{\delta_1}, X_{\gamma'}) \simeq \mathrm{Mor}_{\mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k))}(k \to k, 0 \to k[1]) \simeq 0$$

$$\mathrm{Mor}(L_{\delta_2}, X_{\gamma'}) \simeq \mathrm{Mor}_{\mathrm{Fun}(\Delta^1, \mathcal{D}^{\mathrm{perf}}(k))}(k \to 0, k[1] \to k[1]) \simeq 0\,.$$

Thus $\mathrm{Mor}(X_\gamma, X_{\gamma'}) \simeq 0 \times_{k[1]} 0 \simeq k$. A similar computation shows $\mathrm{Mor}(X_{\gamma'}, X_\gamma) \simeq k[-1]$.

Remark 3.11. The two classes of morphisms in topological Fukaya categories arising from the two types of intersections can be distinguished categorically by their support as follows: Directed boundary intersections give rise to morphisms which evaluate at the corresponding external edge non-trivially. Crossings give rise to morphisms which evaluate trivially at every external edge of the ribbon graph.

4 Lecture 4: Induction and perverse schobers

Let e be an edge of a ribbon graph $\mathbf{G}$ and let $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ be a constructible sheaf. Then restriction of sections of the Grothendieck construction at $e \in \mathrm{Exit}(\mathbf{G})$ defines a functor

$$\mathrm{ev}_e\colon \mathcal{L}(\mathbf{G}, \mathcal{F}) \to \mathcal{F}(e)\,.$$

In the first part of this lecture, we will study this functor and the left adjoint functor $\mathrm{ind}_e^L\colon \mathcal{F}(e) \to R\Gamma(\mathbf{G},\mathcal{F})$ of its restriction to $R\Gamma(\mathbf{G},\mathcal{F})$, called induction (the adjoint of ev_e taking values in $\mathcal{L}(\mathbf{G},\mathcal{F})$ is easy to describe, its simply given by left Kan extension). In the second part, we will consider a broader class of constructible sheaves on graphs, called perverse schobers, containing the gentle sheaves.

4.1 Stop removal

Let $\mathbf{G}$ be a spanning graph of a marked surface and choose an external edge of $\mathbf{G}$. Let $\mathbf{G}'$ be the ribbon graph obtained from removing e. Let v be the vertex incident to e. Given a constructible sheaf $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$, we define the constructible sheaf $\mathcal{F}'\colon \mathrm{Exit}(\mathbf{G}') \to \mathrm{St}$ as follows:

- We set $\mathcal{F}'(v) = \mathcal{F}(v) \times_{\mathcal{F}(e)} 0$ to be the fiber of $\mathcal{F}(v \to e)$. For $f \neq e$ an edge incident to v, we set $\mathcal{F}'(v \to f)$ to be the composite $\mathcal{F}'(v) \to \mathcal{F}(v) \xrightarrow{\mathcal{F}(v\to e)} \mathcal{F}(e)$.
- For all $v \neq x \in \mathrm{Exit}(\mathbf{G}')$, we set $\mathcal{F}'(x) = \mathcal{F}(x)$. All remaining functors in $\mathcal{F}'$ further coincide with the functors in $\mathcal{F}'$.

Proposition 4.1. *(1) There exists a fiber sequence in* St

$$R\Gamma(\mathbf{G}',\mathcal{F}') \hookrightarrow R\Gamma(\mathbf{G},\mathcal{F}) \xrightarrow{\mathrm{ev}_e} \mathcal{F}(e)\,.$$

(2) If $\mathcal{F}$ is gentle, then so is $\mathcal{F}'$.

Proof. Part (1) follows from the fact that limits commute with limits.

Part (2) can be shown using that the fiber of $\pi_{n-2}\colon \mathrm{Fun}(\Delta^{n-2},\mathcal{D}(k)) \to \mathcal{D}(k)$ is equivalent to $\mathrm{Fun}(\Delta^{n-3},\mathcal{D}(k))$. $\square$

Remark 4.2. Proposition 4.1 is the sheaf version of a similar localization sequence for partially wrapped Fukaya categories in the case of marked surface [GPS24]. This localization sequence is referred to as stop removal.

In the setting where every boundary component contains a marked point, the sheaf and cosheaf version of stop removal can be interchanged. As soon as one removes the last external edge e incident to a boundary component, the fiber of ev_e becomes different to the cofiber of ev_e^L: the former will be a proper version of Fukaya category category, the latter the smooth wrapped Fukaya category.

4.2 Induction

Given a gentle sheaf $\mathcal{F}$, we saw that any functor $\mathcal{F}(v \to e)^L$ maps k to a simple S_i, $0 \leq i \leq n-1$, up to shift. Global sections associated with arcs are thus built out p-relative Kan extensions of local objects, the L_δ^v lying at vertices, which arise from even more local objects, namely an object in $\mathcal{F}(e) = \mathcal{D}(k)$. This is the local version of a description of left induction in terms of arcs:

Let $\mathbf{G}$ be a trivalent spanning graph of a marked surface $\mathbf{S}$. Let e be an edge of $\mathbf{G}$. We let $\gamma_e^{\circlearrowright}$ be the arc that, starting at e, is composed of segments (in both directions of e) which always turns right at the trivalent vertex of $\mathbf{G}$.

Proposition 4.3 ([Chr25a, Sections 4.1,4.2])**.** *The functor* $\mathrm{ev}_e\colon R\Gamma(\mathbf{G},\mathcal{F}) \to \mathcal{F}(e)$ *admits a left adjoint, denoted* ind_e^L*, satisfying*

$$\mathrm{ind}_e^L(k) \simeq X_{\gamma_e^{\circlearrowright}} \in R\Gamma(\mathbf{G},\mathcal{F})\,,$$

where $X_{\gamma_e^{\circlearrowright}}$ denotes the global section associated with the arc $\gamma_e^{\circlearrowright}$.

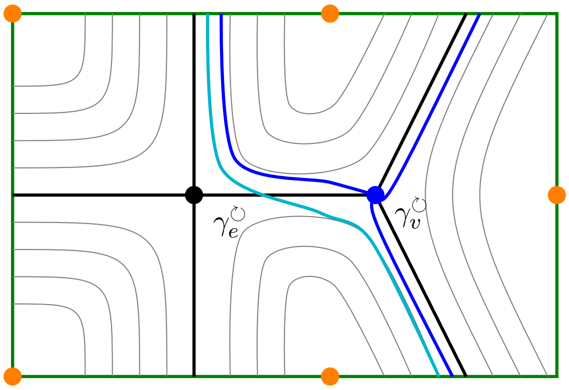


Figure 4: This figure appears as [Chr25a, Figure 2]: the arc $\gamma_e^{\circlearrowright}$ in the 5-gon starts at the middle edge e and always turns right. The broken arc, we call these webs, $\gamma_v^{\circlearrowright}$ is a similar arc starting at the right vertex v. Note that these arcs are, away from their starting points, trajectories for the line field induced by the ribbon graph.

Proof sketch. To simplify the notation, we only prove this in the case that e is external. In this case, γ is the composite of segments $\delta_1, \dots, \delta_n$ with δ_i starting at an edge e_i and ending at an edge e_{i+1} following one step in the clockwise direction, with $e_1 = e$. Let v_i be the vertex near which δ_i lies.

We note that $L^{v_i}_{\delta_i}(v_i) \simeq \mathcal{F}(v_i \to e_i)^R(L_{\delta_i}(e_i))$. Let $Y \in R\Gamma(\mathbf{G}, \mathcal{F})$ be a global section. Then $\mathrm{Mor}_{R\Gamma(\mathbf{G},\mathcal{F})}(X_{\gamma_e^{\circlearrowright}}, Y)$ is equivalent to the limit of the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L_{e_1}, Y) \arrow[dr] & & \mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L_{e_2}, Y) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\
& \mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L^{v_1}_{\delta_1}, Y) & & \mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L^{v_2}_{\delta_2}, Y) & & \mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L^{v_n}_{\delta_n}, Y
\end{tikzcd}$$

where the objects $L_{e_1}, L^{v_2}_{\delta_2}, L^{v_n}_{\delta_n}$ are understood to be suitably shifted.

By construction, $L^{v_i}_{\delta^i}(v_i) \simeq \mathcal{F}(v_i \to e_{i-1})^L(L_{e_{i-1}}(e_{i-1}))$. Since $L^{v_i}_{\delta_i}$ is the left Kan extension of $L^{v_i}_{\delta_i}(v_i)$, we have

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L^{v_i}_{\delta_i}, Y) &\simeq \mathrm{Mor}_{\mathcal{F}(v)}(L^{v_i}_{\delta_i}(v), Y(v)) \\
&\simeq \mathrm{Mor}_{\mathcal{F}(v)}(\mathcal{F}(v_i \to e_{i-1})^L(L_{e_{i-1}}(e_{i-1})), Y(v_i)) \\
&\simeq \mathrm{Mor}_{\mathcal{F}(e)}(L_{e_{i-1}}(e_{i-1}), Y(e_i)) \\
&\simeq \mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L_{e_i}, Y)\,.
\end{aligned}$$

This shows that the right pointed morphisms in the above diagrams are equivalences. The colimit is hence given by

$$\mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L^{v_1}_{\delta_1}, Y) \simeq \mathrm{Mor}_{\mathcal{L}(\mathbf{G},\mathcal{F})}(L_{e_1}, Y) \simeq Y(e_1) = \mathrm{ev}_e(Y) \in \mathcal{D}(k)\,.$$

We thus have

$$\mathrm{Mor}(X_{\gamma_e^{\circlearrowright}}, Y) \simeq \mathrm{Mor}(k, \mathrm{ev}_e(Y))\,,$$

functorial in Y, from which it follows that[6] $X_{\gamma_e^{\circlearrowright}} \simeq \mathrm{ind}_e^L(k)$, as desired. □

[6]Warning: in this conclusion we are being sloppy. With minor additional work and using the characterization of adjunctions from [Cis19, Def. 6.1.3], one can however indeed use this computation to deduce the desired description of the adjoint.

Corollary 4.4. *Let e be an external edge. The functor $\operatorname{ind}_e^L\colon \mathcal{F}(e) = \mathcal{D}^{\mathrm{perf}}(k) \to R\Gamma(\mathbf{G},\mathcal{F})$ is fully faithful if and only if the boundary component contains two marked points.*

If $\operatorname{ind} e^L$ is fully faithful, the fiber sequence from Proposition 4.1.(1) is also a cofiber sequence (and in fact even induces a recollement).

Proof. To prove fully faithfulness, it suffices to show that the unit $\mathrm{id}_{\mathcal{F}(e)} \to \mathrm{ev}_e \operatorname{ind}_e^L$ is an equivalence. Since $\gamma_e^{\circlearrowleft}$ describes the support of $\operatorname{ind}_e^L(k)$, we must simply inspect this arc.

The arc $\gamma_e^{\circlearrowleft}$ is obtained by starting at the boundary component of e, following close to it in the counterclockwise direction until shortly after the next marked point is reached and then ending at the boundary component. Such arcs are also called boundary arcs, they cut out a 1-gon in the boundary. If the boundary component has at least two marked points, then $\gamma_e^{\circlearrowleft}$ has exactly one endpoints near e, and $\mathrm{ev}_e \operatorname{ind}_e^L(k) \simeq k$, as desired. If the boundary component contains a single marked point, then $\gamma_e^{\circlearrowleft}$ also ends near e and $\mathrm{ev}_e \operatorname{ind}_e^L(k) \simeq k \oplus k[n]$ for some $n \in \mathbb{Z}$. In this case, the unit is not an equivalence. □

One can similarly geometrically characterize when ind_e^L is fully faithful. The arising recollements recover known decompositions of topological Fukaya categories, which can be used for instance to compute additive invariants of these categories, see for instance [Dyc17].

4.3 Outlook: perverse schobers

The notion of a categorified perverse sheaf, called perverse schober[7], was proposed by Kapranov–Schechtman [KS14]. The theory on surfaces was initiated in [Chr22] and further worked out in [CHQ23, Chr23, Chr25a].

Definition 4.5. Let $n \geq 1$. A perverse schober on the n-spider consists of the following data:

(1) If $n = 1$, a spherical functor of stable ∞-categories

$$F\colon \mathcal{V} \to \mathcal{N}$$

meaning that F admits a right adjoint G, such that the twist functor $C_{\mathcal{V}} = \mathrm{cof}(\mathrm{id}_{\mathcal{V}} \xrightarrow{\mathrm{unit}} GF) \in \mathrm{Fun}(\mathcal{V},\mathcal{V})$ and cotwist functor $C_{\mathcal{N}} = \mathrm{fib}(FG \xrightarrow{\mathrm{counit}} \mathrm{id}_{\mathcal{N}}) \in \mathrm{Fun}(\mathcal{N},\mathcal{N})$ are autoequivalences.

(2) If $n \geq 2$, a collection of functors of stable ∞-categories

$$(F_i\colon \mathcal{V}^n \longleftrightarrow \mathcal{N}_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$$

satisfying that

(a) F_i admits adjoints $E_i \dashv F_i \dashv G_i \dashv H_i$,

(b) G_i is fully faithful, which is equivalent to $F_iG_i \simeq \mathrm{id}_{\mathcal{N}_i}$ via the counit,

(c) $F_i \circ G_{i+1}$ is an equivalence of ∞-categories,

(d) $F_i \circ G_j \simeq 0$ if $j \neq i, i+1$,

(e) $\mathrm{fib}(H_{i+1}) = \mathrm{fib}(F_i)$ as full subcategories of $\mathcal{V}^n$.

Note that a collection of functors as above is the same data as a functor $\mathrm{Exit}(G_n) \to \mathrm{St}$, with G_n the n-spider, mapping $v \to e_i$ to F_i.

[7] Schober is german for (hay) stack.

Remark 4.6. Let $n \geq 2$ and $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}_n) \to \mathrm{St}$ a perverse schober on the n-spider. Passing to left adjoints from condition (e) implies that $\mathrm{Im}(\mathcal{F}(v \to e_i)^L) = \mathrm{Im}(\mathcal{F}(v \to e_{i+1})^R)$. Thus $\mathcal{F}(v \to e_{i+1} \circ \mathcal{F}(v \to e_i)^L \colon \mathcal{N}_i \to \mathcal{N}_{i+1}$ is an equivalence, satisfying

$$\mathcal{F}(v \to e_i)^L \simeq \mathcal{F}(v \to e_{i+1})^R \circ \mathcal{F}(v \to e_{i+1}) \circ \mathcal{F}(v \to e_i)^L \,.$$

Stated differently, this means that left induction from e_i to v yields the same objects as right induction from e_{i+1} to v. The corresponding objects are described by the boundary arc in the n-gon going from e_i to e_{i+1}.

Definition 4.7. A constructible sheaf $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ is called a $\mathbf{G}$-parametrized perverse schober if for each vertex v of valency n of $\mathbf{G}$, with corresponding inclusion $\mathrm{Exit}(G_n) \subset \mathrm{Exit}(\mathbf{G})$, the restriction $\mathcal{F}|_{\mathrm{Exit}(\mathbf{G}_n)}$ defines a perverse schober on the n-spideri n the sense of Definition 4.5

The notion of a perverse schober categorifies the notion of a perverse sheaf, in the sense that applying K_0 to the diagram describing the perverse schober yields a diagram of vector spaces encoding a perverse sheaf.

Exercise 3. Show that every gentle sheaf defines a perverse schober.

Many features of the theory of gentle sheaves generalizes to the theory of perverse schobers. For instance, one can construct objects from arcs (though typically not all indecomposable global sections will be of this form) and describe the derived Homs between arc objects in terms of intersections. Stop removal and pushforward along contractions of ribbon graphs are also possible for perverse schobers. For induction, we obtain the following:

Proposition 4.8 ([Chr25b])**.** *Let $\mathcal{F}$ be a $\mathbf{G}$-parametrized perverse schober. Let e be an edge of $\mathbf{G}$. Then the functor $\mathrm{ev}_e\colon R\Gamma(\mathbf{G},\mathcal{F}) \to \mathcal{F}(e)$ admits a left adjoint $\mathrm{ind}_e^L, \mathrm{ind}_e^R$, mapping each object $X \in \mathcal{F}(e)$ to a global section whose support is described by $\gamma_e^{\circlearrowright}$.*

The construction of the global section $\mathrm{ind}_e^L(X)$ is analogous to the case of gentle sheaf, it is obtained by gluing together lax sections along $\gamma_e^{\circlearrowright}$. These local sections are all lax Kan extensions of sections of the form $\mathcal{F}(v' \to e')^L(Y)$ with v' a vertex passed by $\gamma_e^{\circlearrowright}$.

The induction from vertices are similarly describes by the branched arcs starting at the vertex, as in Figure 4.

The theory of perverse schobers is inspired by Picard–Lefschetz theory and specifically the construction of Fukaya–Seidel categories [Sei08], see also [Chr23] for a discussion of the relation with the latter. The descent properties of partially wrapped Fukaya categories shown in [GPS24] essentially imply that partially wrapped Fukaya categories of symplectic manfiolds equipped with a Lefschetz fibration to a surface can be computed as the global sections of an induced perverse schober whose edge stalks are the wrapped Fukaya category of the fiber.

References

[CHQ23] M. Christ, F. Haiden, and Y. Qiu. Perverse schobers, stability conditions and quadratic differentials I. arXiv:2303.18249, 2023.

[Chr22] M. Christ. Cluster theory of topological Fukaya categories. arXiv:2209.06595, 2022.

[Chr23] M. Christ. Relative Calabi-Yau structures and perverse schobers on surfaces. arXiv:2311.16597, 2023.

[Chr25a] M. Christ. Induction in perverse schobers and cluster tilting theory. arXiv:2509.01689, 2025.

[Chr25b] M. Christ. ∞-categorical group quotients via skew group algebras. arXiv:2501.13666, 2025.

[Cis19] D. Cisinski. *Higher categories and homotopical algebra.* Cambridge Studies in Advanced Mathematics 180, Cambridge University Press, 2019.

[DK15] T. Dyckerhoff and M. Kapranov. Crossed simplicial groups and structured surfaces. *Stacks and categories in geometry, topology, and algebra, Contemp. Math., 643, Amer. Math. Soc., Providence, RI*, pages 37–110, 2015.

[DK18] T. Dyckerhoff and M. Kapranov. Triangulated surfaces in triangulated categories. *J. Eur. Math. Soc. (JEMS)*, 20(6):1473–1524, 2018.

[Dyc17] T. Dyckerhoff. A1-homotopy invariants of topological Fukaya categories of surfaces. *Compos. Math.*, 153:1673–1705, 2017.

[GPS24] S. Ganatra, J. Pardon, and V. Shende. Sectorial descent for wrapped Fukaya categories. *J. Am. Math. Soc.*, 37(2):499–635, 2024.

[HKK17] F. Haiden, L. Katzarkov, and M. Kontsevich. Flat surfaces and stability structures. *Publ. Math. Inst. Hautes Études Sci,*, 126(1):247–318, 2017.

[KS14] M. Kapranov and V. Schechtman. Perverse Schobers. arXiv:1411.2772, 2014.

[LP20] Y. Lekili and A. Polishchuk. Derived equivalences of gentle algebras via Fukaya categories. *Math. Ann.*, 376(5):187–225, 2020.

[Lur] J. Lurie. Kerodon. `https://kerodon.net`.

[Lur15] J. Lurie. Rotation invariance in algebraic K-theory. *preprint, available* here, 2015.

[Lur17] J. Lurie. *Higher Algebra.* preprint, available here, 2017.

[OPS18] S. Opper, P. Plamondon, and S. Schroll. A geometric model of the derived category of a gentle algebra. arXiv:1801:09659, 2018.

[PT22] M. Porta and J.-B. Teyssier. Topological exodromy with coefficients. arXiv:2211.05004, 2022.

[Sei08] P. Seidel. *Fukaya Categories and Picard-Lefschetz Theory.* European Mathematical Soceity, 2008.