Lecture Notes on Higher Category Theory

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Prerequisites: A good understanding of classical category theory.
Some homotopy theory (Topology I is probably sufficient).
For the last part of the course: familiarity with homological algebra, and in particular the definition of the derived category of an abelian category.

Recommended literature:
Kerodon, Jacob Lurie.
Higher Algebra (chapter 1), Jacob Lurie.

Supplementary reading recommendations:
Categories for the working mathematician, Saunders Mac Lane.
Simplicial Homotopy Theory, John Goress–Paul Goerss.
Higher Categories and Homotopical Algebra, Denis-Charles Cisinski.
An introduction to homological algebra, Charles Weibel.
Lecture notes on homological algebra (esp. chapter 3), Merlin Christ.

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0 Introduction

What is higher category theory?

Higher category theory can be seen as a combination of classical category theory with homotopy theory. There are two complementary viewpoints on what an ∞-category should be:

(1) Higher categories are called so because they have *higher morphisms*. A classical category, henceforth referred to as a 1-category, admits objects and 1-morphisms between these. In a 2-category, there are objects, 1-morphisms and 2-morphisms between 1-morphisms. You have encountered a 2-category already, namely the 2-category with objects 1-categories, functors as 1-morphisms, and natural transformations as 2-morphisms. An ∞-category has n-morphisms for all n. However, these are required to be invertible, starting at level $n = 2$. Because of this, ∞-categories are also referred to a $(\infty, 1)$-categories.

(2) Another viewpoint on ∞-categories is that they are categories enriched in topological spaces. This means one replaces the Hom sets in the category with Hom topological spaces and requires the composition maps to be continuous. However, one would only be interested in the Hom spaces up to homotopy, so an ∞-category would roughly be a topologically enriched category up to homotopy equivalence of the Hom spaces. In this way, each topological space X defines an ∞-category: Objects are points in the space and the space of morphisms from x to y is the space of paths from x to y in X. For instance, the endomorphism space of a point is the based loop space of X.

An $(\infty, 0)$-category is also called an ∞-groupoid, since all its n-morphisms, $n \geq 1$, are invertible. A topological space, considered up to homotopy, is called a homotopy type. The homotopy hypothesis, due to Grothendieck, states the following:

Homotopy hypothesis: ∞-groupoids are equivalent to homotopy types.

We will discuss a formulation of the homotopy hypothesis in the course, modeling ∞-groupoids in terms of simplicial sets. In this way, the theory of homotopy types embeds into the theory of ∞-categories. Higher category theory can thus be seen as a strict generalization of homotopy theory.

On the other hand, each 1-category defines an ∞-category, by declaring all n-morphisms, for $n \geq 2$, to be trivial. This construct will be called the nerve. Higher category theory is thus also a strict generalization of the theory of 1-categories.

What is higher category theory good for?

Studying higher categories comes with many technical difficulties and its results are often inaccessible to non-experts. It is thus particularly important to understand what this theory can be used for, to be able to explain to other mathematicians (or the public) why they should care. Higher category theory has seen remarkable recent applications (to other areas of pure mathematics). The number of problems that have been solved using higher category theory, and for which a solution without it is not available, is quickly rising.

So, to roughly quote Jacob Lurie at the 2010 ICM, higher category theory is theory for the sake of other theory.

We thus wish to highlight three possible ways in which higher categories has been used. During the course, we will further study the second of these applications.

- Higher category theory not only bases on the language of homotopy theory, but has arguably also revolutionized the study of homotopy theory and stable homotopy theory. Higher category theory allows to intrinsically encode homotopical structures without relying on explicit models. Manipulations of the homotopical structures can then be done using universal constructions and universal properties that are homotopy invariant.

- The derived 1-category $D(A)$ of an abelian category A has a conceptual flaw: the cone $\mathrm{cone}(f_*)$ of a chain map f_* in $D(A)$ is not the pushout of the morphism along 0 (in analogy to the cokernel in an abelian category). There exists an ∞-categorical version of $D(A)$, the derived ∞-category $\mathcal{D}(A)$, in which $\mathrm{cone}(f_*)$ has the desired universal property: it is the ∞-categorical pushout. The resolution of this flaw is indicative that the ∞-categorical formulation is conceptually advantageous.
  An axiomatization of the properties of the derived ∞-category $\mathcal{D}(A)$ is the notion of a stable ∞-category, proposed by Lurie in 2006. One can consider the study of stable ∞-categories as the natural setting of homological algebra and its applications.

- The theory of $(\infty, 1)$-categories can be generalized to the theory of (∞, n)-category for some $n \in [0, \infty]$. Such higher categories have received considerable attention for the study of cobordisms and topological field theories. The theory of (∞, n)-categories can in fact be built from the theory of $(\infty, 1)$-categories using enrichments.

The above list does not exhaust the interesting applications of the theory of ∞-categories. Higher category theory is being productively applied to a variety of problems in algebra, algebraic, differential, and symplectic geometry, topology, number theory as well as other areas of mathematics. Before some readers get too excited, we also stress that obviously not all mathematical problems can be magically solved by abstract higher category theory. The role of higher category theory in mathematics may be understood as an extension of the role of 1-category theory in areas involving homotopy theory.

What will actually be covered in this course

The goal will be to cover the foundations of the subjects as well as understand first applications. The three main topics of the course will be the following:

- Simplicial homotopy theory: a formulation of homotopy theory using simplicial sets where homotopy types are modeled as Kan complexes.
- The foundations of higher category theory: this includes mapping spaces, co-Cartesian fibrations, limits and colimits, ...
- The notion of a stable ∞-category and the definition of the derived ∞-category of modules over a ring.

The precise content will change depending on feedback and input from the participants.