Lecture Notes on Homological Algebra (Algebra I)

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Prerequisites: Linear Algebra I+II and Einführung in die Algebra.

Recommended literature:
Homological Algebra, C. Weibel
Categories for the Working Mathematician, S. Mac Lane
Methods of Homological Algebra, S. Gelfand and Y. Manin

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Contents

0 Introduction

Homological algebra is sometimes referred to as *abstract nonsense*. Though it is abstract, it is by no means nonsensical. It can however be challenging to explain what the subject is about in simple terms. In the following, we introduce two of the main players in homological algebra, chain complexes and resolutions, and motivate why one may choose to study these.

Definition 0.1. A chain complex of abelian groups A_* consists of a sequence of abelian groups $(A_n)_{n\in\mathbb{Z}}$ and group homomorphisms $d_n\colon A_n \to A_{n-1}$, $n \in \mathbb{Z}$, called the differentials, satisfying that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$.

We can depict a chain complex A_* as follows:

$$\cdots \to A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \to \ldots$$

We will often write d for d_n. We will also consider partially defined chain complexes, such as chain complexes defined only in positive degrees.

Definition 0.2. Let A_* be a chain complex of abelian groups.

(1) The n-th cycles of A_* are
$$Z_n(A_*) \coloneqq \ker(d_n)\,.$$

(2) The n-th boundaries are
$$B_n(A_*) \coloneqq \operatorname{Im}(d_{n+1})\,.$$

Remark 0.3. Since $d^2 = 0$, there are inclusions $B_n(A_*) \subset Z_n(A_*) \subset A_n$.

Definition 0.4. Let A_* be a chain complex.

(1) The n-th homology of A_* is defined as
$$H_n(A_*) \coloneqq Z_n(A_*)/B_n(A_*) = \ker(d_n)/\operatorname{Im}(d_{n+1})\,.$$

(2) A_* is called exact at position $n \in \mathbb{Z}$ if $B_n(A_*) = Z_n(A_*)$, or equivalently $H_n(A_*) \simeq 0$.

Definition 0.5. A morphism between chain complexes $f_*\colon A_* \to B_*$, also called a chain map, consists of a sequence of morphisms of abelian groups $f_n\colon A_n \to B_n$, $n \in \mathbb{Z}$, satisfying that $d \circ f_n = f_{n-1} \circ d$ for all $n \in \mathbb{Z}$.

We can depict chain maps as follows:

$$\begin{tikzcd}
\ldots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "d"] \arrow[d, "f_{n-1}"] & A_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \ldots \\
\ldots \arrow[r] & B_n \arrow[r, "d"] & B_{n-1} \arrow[r, "d"] & B_{n-2} \arrow[r] & \ldots
\end{tikzcd}$$

Remark 0.6. A chain map $f_*\colon A_* \to B_*$ induces a morphism $H_n(f_*)\colon H_n(A_*) \to H_n(B_*)$ on homology.

Later in the course, we will also consider chain complexes of vector spaces, R-modules, sheaves, or with values in any other abelian category. Such chain complexes naturally arise in many areas of mathematics, such as algebra, topology, algebraic geometry or symplectic topology. We illustrate this for simplicial complexes arising in topology below. The homology groups of chain complexes can contain interesting information. This can be used for instance to construct invariants, allowing to distinguish mathematical objects.

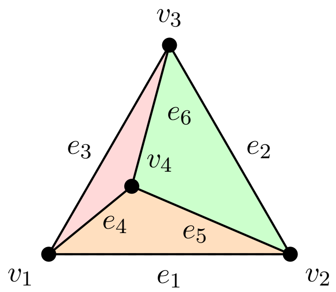
Example: simplicial homology

Definition 0.7. An ordered combinatorial (or abstract) simplicial complex consists of a collection of non-empty subsets K of a finite vertex set $V = \{v_1, \dots, v_N\}$, satisfying that if $\tau \subset \sigma \subset V$ and $\sigma \in K$, then also $\tau \in K$.

A subset $\sigma \in K$ of V with $n+1$ elements is called an n-simplex of K. We denote the set of n-simplicies of K by K_n. Note that $K = \bigcup_{i \geq 0} K_i$.

Remark 0.8. Any combinatorial simplicial complex gives rise to a triangulated polyhedron whose faces of the triangulation are in bijection with the simplices.

Example 0.9. The 2-sphere S^2 is homeomorphic to the following triangulated polyhedron:



The corresponding combinatorial simplicial complex K is given by

$$\begin{aligned} K_0 &= \{v_1, \dots, v_4\} \\ K_1 &= \{e_1, \dots, e_6\} \\ K_2 &= \{F_1, \dots, F_4\} \\ K_i &= \emptyset, \quad \text{for } i \geq 3\,, \end{aligned}$$

where $F_1, \dots, F_4$ are the four faces (one lying at the back).
For instance, the face $F_1 \in K_2$ is given by the set $\{v_1, v_2, v_3\}$ and the edge $e_1 \in K_1$ is given by $\{v_1, v_2\}$. The notation for the 0-simplicies is slightly abusive, as we do not distinguish between v_i and $\{v_i\}$.

Note that an n-simplex $\sigma \in K_n$ is given by an ordered set

$$\sigma = \{v_{i_0}, v_{i_1}, \dots, v_{i_n}\}$$

with $1 \leq i_0 < i_1 < \dots < i_n \leq N$. For $0 \leq j \leq n$, we define the j-th face of σ as

$$\partial_j(\sigma) = \sigma \backslash \{v_{i_j}\} = \{v_{i_0}, \dots, v_{i_{j-1}}, v_{i_{j+1}}, \dots, v_{i_n}\} \in K_{n-1}\,.$$

Definition 0.10. Let K be a directed combinatorial simplicial complex. We define $C_*(K)$ as the following chain complex:

- We set
$$C_n(K) = \mathbb{Z}^{\oplus K_n}$$
for all $n \geq 0$. We can thus consider each n-simplex $\sigma \in K_n$ as an element $\sigma \in C_n(k)$, and any element of $C_n(K)$ can be written as a $\mathbb{Z}$-linear sum of these.

- The differential
$$d\colon C_n(K) \to C_{n-1}(K)$$
is the unique morphism of abelian groups satisfying
$$d(\sigma) = \sum_{0 \leq j \leq n} (-1)^j \partial_j(\sigma)\,.$$

The i-th simplicial homology of K is defined as
$$H_i(K) \coloneqq H_i(C_*(K))\,.$$

Remark 0.11. The terminology for cycles and boundaries in chain complexes can be motivated through the above definition. For instance, for K the combinatorial simplicial complex from Example 0.9, the boundary of the face F_1 is given by $d(F_1) = \partial_0(F_1) - \partial_1(F_1) + \partial_2(F_2) = e_2 - e_3 + e_1$. Each boundary is also a cycle, and indeed $e_2 - e_3 + e_1$ is geometrically a cycle.

Exercise 0.12. Let K be the combinatorial simplicial complex from Example 0.9. Then
$$H_i(K) \simeq \begin{cases} \mathbb{Z} & i = 0, 2 \\ 0 & \text{else.} \end{cases}$$

In general, the i-th homology of a combinatorial simplicial complex can be loosely thought of as a count of i-dimensional holes of the corresponding topological space.

Resolutions and Ext^1

A chain complex is called exact if it is exact in every position. Exact chain complexes are sometimes also referred to as exact sequences.

Short exact sequences

Example 0.13. A 3-term chain complex A_* consists of a chain complex with non-zero entries concentrated in degrees $2, 1, 0$:
$$\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to \ldots\,.$$

We have $B_2(A_*) = \mathrm{Im}(0 \to A) = 0$ and thus $H_2(A_*) = Z_2(A_*) = \ker(f)$. Therefore, it is exact in position 2 if and only if f is injective, i.e. an inclusion of abelian groups.

Similarly, we have $C = Z_0(A_*) = \ker(C \to 0)$ and thus $H_0(A_*) = C/\operatorname{Im}(g)$. Thus, it is exact in position 0 if g is surjective.

Suppose the 3-term chain complex is exact in positions 2 and 0. We claim that A_* is exact in position 1 if and only if the arising morphism $B/A \to C$ is an isomorphism of abelian groups.

We first note that $\operatorname{Im}(f) = B_1(A_*) \simeq A$ since f is injective.

$\Longrightarrow$: If $B/A \to C$ is an isomorphism, then by the commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "g"] \arrow[d, two heads] & C \\ & B/A \arrow[ur, "\simeq"'] & \end{tikzcd}$$

we find that $\ker(g) \simeq A$, and thus $\operatorname{Im}(f) = \ker(g)$.

$\Longleftarrow$: Suppose that $\ker(g) = \operatorname{Im}(f) \simeq A$. Since $g\colon B \to C$ is surjective, it is a quotient morphism, meaning that the induced morphism $B/\ker(g) \to C$ is an isomorphism. We thus have

$$C \simeq B/\ker(g) = B/\operatorname{Im}(g) \simeq B/A\,.$$

Definition 0.14. A short exact sequence of abelian groups consists of an exact 3-term chain complex

$$0 \to A \to B \to C \to 0\,.$$

In the case, we also call B an extension of C and A.

Example 0.15. For any $n \geq 1$, the sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$$

is short exact.

Resolutions

Definition 0.16. Let M be an abelian group. A left resolution of M consists of a chain complex A_* in positive degrees together with a morphism $\epsilon\colon A_0 \to M$, satisfying that the augmented chain complex

$$\ldots \xrightarrow{d} A_2 \xrightarrow{d} A_1 \xrightarrow{d} A_0 \xrightarrow{\epsilon} M \to 0$$

is exact. The morphism ϵ is also called the augmentation map. If it exists, the minimal number $n \geq 0$ with $A_N \simeq 0$ for all $N \geq n$ is called the length of the resolution.

Definition 0.17. Let M be an abelian group. A free resolution of M consists of a resolution A_* of M, such that A_n is a free abelian group (i.e. equivalent to $\mathbb{Z}^{\oplus I}$ for some set I) for all $n \geq 0$.

Example 0.18. The chain complex $\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$ of length 2.

Exercise 0.19. Every finitely generated abelian group admits a free resolution of length 2.

Remark 0.20. If A_* is a resolution of an abelian group M, then

$$H_n(A_*) \simeq \begin{cases} M & n = 0 \\ 0 & n \in \mathbb{Z}, n \neq 0\,. \end{cases}$$

Given two abelian groups A, B, we denote by $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ the abelian group of group homomorphisms from A to B.

Exercise 0.21. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Then for any abelian group X, the sequence

$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(C, X) \to \mathrm{Hom}_{\mathrm{Ab}}(B, X) \to \mathrm{Hom}_{\mathrm{Ab}}(A, X)$$

is exact.

Using free resolutions, we will be able to measure the failure of the above sequence to be exact when extended with 0 to the right and to access 'hidden homological information':

Example 0.22. There is an isomorphism

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \simeq 0\,.$$

Starting with the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

we thus obtain the sequence

$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}) \to 0$$

isomorphic to sequence

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0 \to \dots$$

which is not exact at position 4. There is a canonical guess on how to make the above sequence exact, namely by adding $\mathbb{Z}/2\mathbb{Z}$ at position 5 (counted from the left):

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0 \to \dots$$

The group $\mathbb{Z}/2\mathbb{Z}$ computes the extension group $\mathrm{Ext}^1_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$:

For two abelian groups M, B, the group $\mathrm{Ext}^1_{\mathrm{Ab}}(M, B)$ is defined as the first homology of the chain complex obtained from applying $\mathrm{Hom}_{\mathrm{Ab}}(\text{-}, B)$ to any free resolution A_* of M. This homology group turns out to be independent of the choice of free resolution.

Later in the course, we will systematically study these Ext groups (and other derived functors) and prove the following:

Theorem 0.23. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and X an abelian group. Then there exists an exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathrm{Ab}}(C,X) \arrow[r] & \mathrm{Hom}_{\mathrm{Ab}}(B,X) \arrow[r] & \mathrm{Hom}_{\mathrm{Ab}}(A,X) \arrow[dll] \\ & \mathrm{Ext}^1_{\mathrm{Ab}}(C,X) \arrow[r] & \mathrm{Ext}^1_{\mathrm{Ab}}(B,X) \arrow[r] & \mathrm{Ext}^1_{\mathrm{Ab}}(A,X) \arrow[r] & 0 \end{tikzcd}$$

Furthermore, we will find that $\mathrm{Ext}^1_{\mathrm{Ab}}(C,X)$ counts the (equivalence classes of) short exact sequences (i.e. extensions) of the form $0 \to X \to E \to C \to 0$.

1 Abelian categories

This sections introduces the basic language and notions used in the theory of homology algebra. A central notion is that of an abelian category, whose main example is given by the category of modules over a ring. Along the way, we introduce central notions from category theory, such as (co)limits and adjunctions.

1.1 Modules over rings

Definition 1.1.1.

(1) A unital, associative ring $(R,+,\cdot)$ consists of a set R together with two binary operations
$$+,\ \cdot : R \times R \to R$$
satisfying the following:

- $(R,+)$ defines an abelian group with neutral element denoted by 0_R or 0.
- The multiplication $\cdot$ satisfies associativity, meaning that
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
for all $a,b,c \in R$, and distributivity with respect to $+$, meaning that
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
and
$$(a + b) \cdot c = a \cdot c + b \cdot c\,.$$
for all $a,b,c \in R$.
- There exists a multiplicative unit $1 = 1_R \in R$ satisfying $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

All rings will be assumed to be unital and associative. We will also write ab for $a \cdot b$.

(2) A ring R is called commutative if $a \cdot b = b \cdot a$ for all $a, b \in R$.

(3) A field is a commutative ring satisfying that $(R \backslash \{0\}, \cdot)$ defines an abelian group with neutral element 1.

Definition 1.1.2. Let R, R' be two rings. A map between the underlying sets $f \colon R \to R'$ is called a unital ring homomorphism if

- $f(a+b) = f(a) + f(b)$ for all $a, b \in R$,
- $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in R$,
- and $f(1_R) = 1_{R'}$.

As with rings, we will assume all ring homomorphisms to be unital.

Example 1.1.3. For each ring R, there exists a unique ring homomorphism $\mathbb{Z} \to R$.

Definition 1.1.4. Let R be a ring.

(1) A left module M over R consists of an abelian group $(M, +)$ together with a left R-action
$$\cdot \colon R \times M \to M$$
satisfying that
- $(a+b) \cdot m = a \cdot m + b \cdot m$ and $a \cdot (m + m') = a \cdot m + a \cdot m'$ for all $a, b \in R$, $m, m' \in M$,
- $a \cdot (b \cdot m) = (a \cdot b) \cdot m$ for all $a, b \in R$ and $m \in M$,
- and $1_R \cdot m = m$ for all $m \in M$.

(2) A right module M over R consists of an abelian group $(M, +)$ together with a right R-action
$$\cdot \colon M \times R \to M$$
satisfying that
- $m \cdot (a+b) = m \cdot a + m \cdot b$ and $(m + m') \cdot a = m \cdot a + m' \cdot a$ for all $a, b \in R$, $m, m' \in M$,
- $(m \cdot a) \cdot b = m \cdot (a \cdot b)$ for all $a, b \in R$ and $m \in M$,
- and $m \cdot 1_R = m$ for all $m \in M$.

In the following, we will primarily work with left R-modules, and for brevity will refer to left R-modules simply as R-modules.

Example 1.1.5. (1) Let k be a field. A left k-module amounts to a k-vector space.

(2) Let $I \subset R$ be a left ideal of a ring. Then I is a left R-module with the action given by multiplication in R. In particular, each ring R is a left module over itself.

(3) The trivial abelian group 0 admits a unique R-module structure for each ring R.

(4) Each abelian group M admits a unique $\mathbb{Z}$-module structure: By distributivity, the action of $n \in \mathbb{Z}$ on $x \in M$ must be given by

$$n \cdot x = \underbrace{(1 + \cdots + 1)}_{n\text{-times}} \cdot x = \underbrace{x + \cdots + x}_{n\text{-times}},$$

where the latter addition is the addition in M. One readily checks that this action indeed defines a $\mathbb{Z}$-module structure on M.

Definition 1.1.6. Let R be a ring.

(1) Let M, N be left R-modules. A left R-module homomorphism $f\colon M \to N$ consists of a group homomorphism satisfying that

$$f(a \cdot m) = a \cdot f(m)$$

for all $a \in R$ and $m \in M$.

(2) An isomorphism between left R-modules consists of a bijective R-module homomorphism $f\colon M \to N$. We also write $f\colon M \simeq N$.

(3) Given two left R-modules M, N, we denote by $\mathrm{Hom}_R(M, N)$ the set of left R-module homomorphisms $f\colon M \to N$.

Note that each R-module isomorphisms admits a unique two-sided inverse R-module homomorphism.

Remark 1.1.7. Given two R-modules M, N, the set $\mathrm{Hom}_R(M, N)$ admits the structure of an abelian group, with addition of $f, g \in \mathrm{Hom}_R(M, N)$ defined as the R-module homomorphism

$$f + g\colon M \to N, \quad m \mapsto f(m) + g(m).$$

Example 1.1.8. Let R be a ring, I a set, and $\{M_i\}_{i \in I}$ an I-indexed family of R-modules.

(1) The product $\prod_{i \in I} M_i$ is given by the R-module with

- elements given by I-indexed tuples $(m_i)_{i \in I}$, with $m_i \in M_i$ for all $i \in I$,
- the addition of these I-indexed tuples defined componentwise, meaning that
$$(m_i)_{i \in I} + (m'_i)_{i \in I} = (m_i + m'_i)_{i \in I},$$
this yielding an abelian group structure on $\prod_{i \in I} M_i$ with neutral element $(0_{M_i})_{i \in I}$,

- and left R-action defined componentwise, meaning as

$$a \cdot (m_i)_{i\in I} = (a \cdot m_i)_{i\in I}$$

for any $a \in R$.

(2) The I-fold direct sum $\bigoplus_{i\in I} M_i$ can be described similarly, with the difference that its elements are I-indexed tuples $(m_i)_{i\in I}$, with $m_i \in M_i$ for all $i \in I$, satisfying that the set $\{i \in I|\ m_i \neq 0_{M_i}\}$ is finite. There is an apparent inclusion of R-modules

$$\bigoplus_{i\in I} M_i \subset \prod_{i\in I} M_i\,,$$

which is an equality if I is finite.

Definition 1.1.9. An R-module M is called free if there exists a set I and an R-module isomorphism $M \simeq \bigoplus_I R$. We also write $R^{\oplus I}$ for $\bigoplus_I R$.

Proposition 1.1.10 (The universal property of free R-modules)**.** *Let R be a ring, M an R-module and I a set. Then there exists a bijection of sets*

$$\mathrm{Hom}_R(R^{\oplus I}, M) \simeq \mathrm{Hom}_{\mathrm{Set}}(I, M)\,.$$

Proof. We define inverse morphisms of sets

$$\phi\colon\ \mathrm{Hom}_R(R^{\oplus I}, M) \longleftrightarrow \mathrm{Hom}_{\mathrm{Set}}(I, M) : \psi\,.$$

There is an inclusion of sets $\iota\colon I \to R^{\oplus I}$, defined by mapping $i \in I$ to the I-indexed tuple $(a_j)_{j\in I}$ with $a_i = 1 \in R$ and $a_j = 0 \in R$ if $j \neq i$.

Given an R-module homomorphism $f : R^{\oplus I} \to M$, we set $\phi(f) = f \circ \iota$. Given a morphism of sets $g\colon I \to M$, we define $\psi(g)\colon R^{\oplus I} \to M$ by $\psi(g)((a_j)_{j\in I}) = \sum_{j\in I} a_j \cdot g(j)$. Note that the latter sum in M has only finitely many non-zero terms and is thus defined.

For $f\colon R^{\oplus I} \to M$ and $(a_j)_{j\in I} = \sum_{j\in I} a_j \cdot \iota(j) \in R^{\oplus I}$, we have

$$\begin{aligned} \psi \circ \phi(f)((a_i)_{i\in I}) &\overset{\text{def. of } \psi}{=} \sum_{j\in I} a_j \cdot \phi(f(j)) \\ &\overset{\text{def. of } \phi}{=} \sum_{j\in I} a_j \cdot f(\iota(j)) \\ &= f(\sum_{j\in I} a_j \cdot \iota(j)) \\ &= f((a_j)_{j\in I})\,. \end{aligned}$$

and thus $\psi \circ \phi(f) = f$.

We further have for $g\colon I \to M$ and $j \in I$

$$\phi \circ \psi(g)(j) = \psi(g)(\iota(j)) = g(j)$$

and thus $\phi \circ \psi(g) = g$. This indeed shows that ϕ, ψ are inverse to each other. □

1.2 Categories

Definition 1.2.1. A category $\mathcal{C}$ consists of

- a class $\mathrm{ob}(\mathcal{C})$ of objects,
- for all pairs of objects x, y a set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of morphisms,
- for all triples of objects x, y, z a composition map

$$- \circ - \colon \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)\,,$$

- and for all objects x an identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$,

satisfying associativity, meaning that

$$(h \circ g) \circ f = h \circ (g \circ f)$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(x, y), g \in \mathrm{Hom}_{\mathcal{C}}(y, z), h \in \mathrm{Hom}_{\mathcal{C}}(z, w)$, and unitality, meaning that

$$\mathrm{id}_y \circ f = f = f \circ \mathrm{id}_x$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

If x is an object of a category $\mathcal{C}$, we also write $x \in \mathcal{C}$. If $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, we also write $f \colon x \to y$.

Definition 1.2.2. A morphism $f \colon x \to y$ in a category $\mathcal{C}$ is called an isomorphism if it admits a two-sided inverse, i.e. there exists a morphism $g \colon y \to x$ with $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$.

Definition 1.2.3. A category is called small if its objects form a set.

Definition 1.2.4. Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{C}' \subset \mathcal{C}$ consists of a category $\mathcal{C}'$ with

- $\mathrm{ob}(\mathcal{C}')$ a subclass of $\mathrm{ob}(\mathcal{C})$,
- $\mathrm{Hom}_{\mathcal{C}'}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)$ for all objects x, y of $\mathcal{C}'$,
- and composition of morphisms in $\mathcal{C}'$ given by composition in $\mathcal{C}$.

Example 1.2.5. (1) Each set I defines a category with objects the elements of I and morphisms given by

$$\mathrm{Hom}_I(x, y) = \begin{cases} \{\mathrm{id}_x\} & x = y \\ \emptyset & x \neq y\,. \end{cases}$$

(2) The category Set has as objects the class of all sets and as morphisms functions between sets. The composition law is the usual composition of functions. Given a set X, the identity morphism $\mathrm{id}_X \colon X \to X$ is the identity function, given by mapping any element $x \in X$ to itself.

(3) The category Grp of groups has as objects groups and as morphisms the group homomorphisms.

(4) The category Ab of abelian groups is defined as the full subcategory of Grp whose objects are abelian groups.

(5) Let R be a ring. The category LMod_R of left R-modules has as objects left R-modules and as morphisms the left R-module homomorphisms.

(6) There is a category Ring of unital, associative rings and unital ring homomorphisms.

(7) Let $\mathcal{C}, \mathcal{D}$ be small categories. Then there exists an apparent category $\mathcal{C} \times \mathcal{D}$ with objects $\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = i\,\mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$ and for all pairs of objects $(x, y), (x', y') \in \mathcal{C} \times \mathcal{D}$
$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((x, y), (x', y')) = \mathrm{Hom}_{\mathcal{C}}(x, x') \times \mathrm{Hom}_{\mathcal{D}}(y, y')\,.$$

Definition 1.2.6. Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, morphism sets

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)\,,$$

and composition of $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y)$, $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(y, z)$ is defined as

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f \coloneqq f \circ_{\mathcal{C}} g \in \mathrm{Hom}_{\mathcal{C}}(z, x) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, z)\,.$$

Definition 1.2.7. Let $\mathcal{C}, \mathcal{D}$ be a categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of

- the assignment of an object $F(x) \in \mathcal{D}$ to each object $x \in \mathcal{C}$,
- for each pair $x, y \in \mathcal{C}$ a morphism of sets

  $$F\colon\ \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$

  satisfying that
  $$F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$$
  and
  $$F(g \circ f) = F(g) \circ F(f)$$
  for all morphisms $f\colon x \to y$ and $g\colon y \to z$.

Example 1.2.8.

(1) Let $\mathcal{C}$ be a category. There is the identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$, given by the identity on objects and morphisms of $\mathcal{C}$.

(2) Any functor $F\colon \mathcal{C} \to \mathcal{D}$ determines an opposite functor $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ given by the same assignments on objects and morphisms.

(3) The inclusion of a full subcategory $\mathcal{C}' \subset \mathcal{C}$ defines a functor.

(4) There is a forgetful functor Grp $\to$ Set assigning to each group its underlying set and to each group homomorphism the underlying morphism of sets.

(5) There is a forgetful functor $\mathrm{LMod}_R \to \mathrm{Ab}$, mapping each R-module M to the underlying abelian group.

(6) Let $\mathcal{C}$ be a category and $x \in \mathcal{C}$. Then there is a functor

$$\mathrm{Hom}_{\mathcal{C}}(x, \text{-}) \colon \mathcal{C} \to \mathrm{Set}$$

mapping $y \in \mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$. Similarly, there is a functor

$$\mathrm{Hom}_{\mathcal{C}}(\text{-}, x) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \, .$$

(7) Let R be a ring and M a left R-module. Then there are functors

$$\mathrm{Hom}_R(M, \text{-}) \colon \mathrm{LMod}_R \to \mathrm{Ab}$$

and

$$\mathrm{Hom}_R(\text{-}, M) \colon \mathrm{LMod}_R^{\mathrm{op}} \to \mathrm{Ab} \, .$$

1.3 Limits and colimits

The notion of a limit in a category allows to characterize set-theoretic constructions (such as products, unions, quotients, ...) in terms of universal properties. These clarify the properties of the set-theoretic constructions. However more importantly, they allow to generalize such constructions to all categories at the same time, as well as enable us to consider the relation of the constructions in different categories.

1.3.1 Products and coproducts

Definition 1.3.1. Let $\mathcal{C}$ be a category and $x, y \in \mathcal{C}$ two objects. An object $x \times y \in \mathcal{C}$ together with two morphisms $p_x \colon x \times y \to x$ and $p_y \colon x \times y \to y$ is called the (binary) product of x and y if, for every object $z \in \mathcal{C}$ together with maps $q_x \colon z \to x$ and $q_y \colon z \to y$, there exists a unique morphism $\phi \colon z \to x \times y$ satisfying that

$$q_x = p_x \circ \phi \, , \qquad q_y = p_y \circ \phi \, . \tag{1}$$

We can illustrate Definition 1.3.1 in terms of a commutative diagram:

$$\begin{tikzcd}
 & z \arrow[dl, bend right, "q_y"'] \arrow[d, dashed, "\exists!\phi"] \arrow[dr, bend left, "q_x"] & \\
 & x \times y \arrow[dl, "p_x"] \arrow[dr, "p_y"'] & \\
x & & y
\end{tikzcd}$$

If a binary product exists, the universal property uniquely characterizes it up to unique isomorphism in $\mathcal{C}$ (see below). We will slightly abuse language and speak of *the* product of x and y in $\mathcal{C}$.

Example 1.3.2. Let I, J be two sets. Then the Cartesian product $I \times J$ is given by the set whose elements are tuples (i, j) with $i \in I$ and $j \in J$. We define the projection maps

$$p_I : I \times J \to I, \quad (i, j) \mapsto i$$

and

$$p_J : I \times J \to J, \quad (i, j) \mapsto j\,.$$

We claim that $I \times J$ together with p_I, p_J describes the product of I and J in the category Set.

Let Z be a set together with morphisms $q_I : Z \to I$ and $q_J : Z \to J$. Let $\phi: Z \to I \times J$ be a morphism. For each element $z \in Z$, we can write $\phi(z) = (i_z, j_z)$ for two elements $i_z \in I$ and $j_z \in J$. The equations (1) amount to the conditions

$$q_I(z) = i_z\,, \qquad q_J(z) = j_z\,.$$

We thus see that $\phi: Z \to I \times J$ satisfies (1) if and only if $(i_z, j_z) = (q_I(z), q_J(z))$ for all elements $z \in Z$. The unique morphism ϕ is thus given by the assignment $\phi: z \mapsto (q_I(z), q_J(z))$, showing that $I \times J$ is indeed the limit.

We can generalize binary products to arbitrary products as follows:

Definition 1.3.3. Let $\mathcal{C}$ be a category, K a set, and $(x_k)_{k\in K}$ a K-indexed family of objects in $\mathcal{C}$. Then the product $\prod_{k\in K} x_k$ is any object equipped with morphisms $(p_l : \prod_{k\in K} x_k \to x_k)_{l\in K}$ such that, for any other object $z \in \mathcal{C}$ and morphisms $(q_l : z \to x_l)_{l\in K}$, there exists a unique morphism $\phi: z \to \prod_{k\in K} x_k$ with $q_l = p_l \circ \phi$ for all $l \in K$.

Dual to the notion of a product is that of a coproduct.

Definition 1.3.4. Let $\mathcal{C}$ be a category, K a set, and $(x_k)_{k\in K}$ a K-indexed family of objects in $\mathcal{C}$. Then the coproduct $\coprod_{k\in K} x_k$ is any object equipped with morphisms $(i_l : x_l \to \coprod_{k\in K} x_k)_{l\in K}$ such that for any other object $z \in \mathcal{C}$ and morphisms $(j_l : x_l \to z)_{l\in K}$ there exists a unique morphism $\phi: \coprod_{k\in K} x_k \to z$ with $\phi \circ i_l = j_l$ for all $l \in K$.

We can again illustrate the universal property of a binary coproduct as follows:

$$\begin{tikzcd}
x \arrow[dr, "i_x"] \arrow[ddr, bend right, "j_x"'] & & y \arrow[dl, "i_y"'] \arrow[ddl, bend left, "j_y"] \\
& x \times y \arrow[d, dashed, "\exists!\phi"] & \\
& z &
\end{tikzcd}$$

Example 1.3.5. (1) Products and coproducts in Set are given by products and disjoint unions.

(2) Let R be a ring. Products and coproducts in LMod_R are given by products and direct sums, as in Example 1.1.8.

(3) Products in Grp are given by products of groups, and coproducts of groups are given by the free product.

(4) Products in the category Ring are given by the product of rings. The category of fields does not admit products.

A special case of (co)products are (co)products over the empty set:

Definition 1.3.6. Let $\mathcal{C}$ be a category.

(1) An object $*$ of $\mathcal{C}$ is called terminal if it is the empty product. This means that for every object z of $\mathcal{C}$ there exists a unique morphism $\phi\colon z \to *$.

(2) An object $\emptyset$ of $\mathcal{C}$ is called initial if it is the empty coproduct. This means that for every object z of $\mathcal{C}$ there exists a unique morphism $\emptyset \to z$.

(3) An object X of $\mathcal{C}$ is called a zero object if it is both initial and terminal. We denote zero objects by $0_{\mathcal{C}}$ or 0. If $\mathcal{C}$ admits a zero object, the category $\mathcal{C}$ is called pointed.

Example 1.3.7.

(1) The category Set admits a terminal object $*$, the singleton set, and an initial object $\emptyset$, the empty set.

(2) The category Set_* of pointed sets has as objects sets I equipped with a morphism $* \to I$, and morphisms are morphisms of sets $I \to J$, such that the diagram

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ I \arrow[rr] & & J \end{tikzcd}$$

commutes. The category Set_* is pointed, with zero object the singleton $*$ with the identity morphism $* \to *$.

(3) For any ring R, the category LMod_R is pointed, with 0_{LMod_R} the unique R-module with underlying abelian group 0.

(4) The category Ring admits an initial object, given by $* = \mathbb{Z}$, and a terminal object, given by the zero ring $\emptyset = 0$.

1.3.2 Kernels and cokernels

Given a pointed category $\mathcal{C}$, we call a morphism $f\colon x \to y$ a zero morphism, and write $f = 0$, if f factors as $x \to 0 \to y$. Note that by the universal properties of a zero object, there exists at most one such factorization.

Definition 1.3.8. Let $\mathcal{C}$ be a pointed category and $f\colon x \to y$ a morphism in $\mathcal{C}$.

(1) The kernel of f is any object $\ker(f) \in \mathcal{C}$ together with a morphism $i\colon \ker(f) \to x$, satisfying $f \circ i = 0$ and that for any other morphism $g\colon z \to x$ with $f \circ g = 0$ there exists a unique morphism $\phi\colon z \to \ker(f)$ satisfying that $g = i \circ \phi$.

(2) The cokernel of f is any object $\operatorname{coker}(f) \in \mathcal{C}$ together with a morphism $p\colon y \to \operatorname{coker}(f)$, satisfying $p \circ f = 0$ and that for any other morphism $g\colon y \to z$ with $g \circ f = 0$ there exists a unique morphism $\phi\colon \operatorname{coker}(f) \to z$ satisfying that $g = \phi \circ p$.

$$\begin{tikzcd}
 & z \arrow[dl, dashed, "\exists!\phi"'] \arrow[d, "g"'] \arrow[dd, bend left, "0"] \\
\ker(f) \arrow[r, "i"'] \arrow[dr, "0"'] & x \arrow[d, "f"'] \\
 & y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "f"'] \arrow[dr, "0"'] \arrow[rr, bend left, "0"] & y \arrow[r, "g"'] \arrow[d, "p"] & z \\
 & \operatorname{coker}(f) \arrow[ur, dashed, "\exists!\phi"'] &
\end{tikzcd}$$

Figure 1: The universal properties of the kernel and cokernel.

Example 1.3.9. (1) A kernel of a morphism of abelian groups $f\colon A \to B$ is given by the full subgroup of A with underlying set $A' = \{a \in A | f(a) = 0\}$ together with the inclusion $i\colon A' \subset A$.

(2) The cokernel of a morphism of abelian groups $f\colon A \to B$ is given by the quotient abelian group $B/\operatorname{Im}(f)$, where $\operatorname{Im}(f) \subset B$ is the subgroup given by the image of f. Elements of the quotient abelian group $B/\operatorname{Im}(f)$ are equivalence classes in B under the relation $b \sim b'$ if $b - b' \in \operatorname{Im}(f)$. The morphism $p\colon B \to B/A$ is the quotient map, which maps $b \in B$ to its equivalence class $[b] \in B/\operatorname{Im}(f)$.

Proposition 1.3.10. *Let R be a ring. The category* LMod_R *admits kernels and cokernels. Furthermore, the forgetful functor* $\mathrm{LMod}_R \to \mathrm{Ab}$ *preserves kernels and cokernels.*

Proof. Let $f\colon M \to N$ be a morphism of R-modules. Let $M' \subset M$ be the abelian subgroup with elements $M' = \{m \in M | f(m) = 0\}$. Let $a \in R$ and $m \in M'$. Then $f(a.m) = a.f(m) = 0$, since f is a morphism of R-modules, and thus $a.m \in M'$. We thus see that the R-action $R \times M \to M$ restricts to an R-action $R \times M' \to M'$ on M'. The inclusion $i\colon M' \subset M$ is clearly a morphism between R-modules.

It remains to verify the universal property of the kernel: Let $Z \in \mathrm{LMod}_R$ and $g\colon Z \to M$ a morphism with $f \circ g = 0$. By the universal property of the kernel in Ab, there exists a unique morphism of abelian groups $\phi\colon Z \to M'$ such that $i \circ \phi = g$. We show that ϕ is a morphism of R-modules. Since $i(m) = m$ for all $m \in M'$, we find that $g(z) = \phi(z)$ for all $z \in Z$. We thus have

$$\phi(a.z) = g(a.z) = a.g(z) = a.\phi(z)$$

for all $a \in R$ and $z \in Z$. Since ϕ is the desired morphism of R-modules, as well as the unique morphism of abelian groups, we find that ϕ is the unique morphism of R-modules, showing the universal property.

We next describe the cokernel of f. We first show that the quotient abelian group $N/\operatorname{Im}(f)$ admits an R-action defined by $a \cdot [n] = [a \cdot n]$ for any $a \in R$ and $n \in N$. To show that this action is well defined, let $[n] = [n'] \in N/\operatorname{Im}(f)$. Then $n - n' = f(m)$ for some $m \in M$, and thus

$$a \cdot n - a \cdot n' = a \cdot (n - n') = a \cdot f(m) = f(a \cdot m) \in \operatorname{Im}(f) \,,$$

showing that $[a.n] = [a.n']$ for all $a \in R$. We next verify that the quotient morphism $p\colon N \to N/\operatorname{Im}(f), n \mapsto [n]$ is an R-module homomorphism: We have $p(a \cdot n) = [a \cdot n] = a \cdot [n]$ for all $n \in N$ and $a \in R$.

Finally, it remains to verify the universal property of the cokernel. Let $g\colon N \to Z$ be an R-module homomorphism with $g \circ f = 0$. By the universal property of the cokernel of abelian groups, there exists a unique morphism of abelian groups $\phi\colon N/\operatorname{Im}(f) \to Z$ satisfying that $g = \phi \circ p$. As above, it remains to show that ϕ is an R-module homomorphism. Let $a \in R$ and $n \in N$. Then

$$\phi(a \cdot [n]) = \phi([a \cdot n]) = \phi(p(a \cdot n)) = g(a \cdot n) = a \cdot g(n) = a \cdot \phi(p(n)) = a \cdot \phi([n]) \,,$$

as desired. □

Definition 1.3.11. Let N be an R-module and $M \subset N$ an abelian subgroup. We call M a submodule of N if the inclusion $M \subset N$ is an R-module homomorphism.

We can define quotient modules as cokernels along inclusions:

Definition 1.3.12. Let N be an R-module and $M \subset N$ a submodule. Then the quotient R-module N/M is defined as the cokernel of the inclusion $M \subset N$.

Note that the abelian group underlying the quotient module N/M is the quotient abelian group.

1.3.3 General limits

We next generalize the constructions of products and coproducts and of kernels and cokernels, to general limits.

Given two categories $K, \mathcal{C}$, a K-indexed diagram in $\mathcal{C}$ is defined to be a functor $K \to \mathcal{C}$.

Definition 1.3.13. Let $F\colon K \to \mathcal{C}$ be a K-indexed diagram.

1. A cone $(X, \{p_k\}_{k \in K})$ from X to F consists of an object X of $\mathcal{C}$ together with morphisms $p_k\colon X \to F(k)$ for all $k \in K$ satisfying that

$$p_l = F(\alpha) \circ p_k$$

for all morphisms $\alpha\colon k \to l$ in K.

A morphism of cones $\phi\colon (X, \{p_k\}_{k\in K}) \to (Y, \{q_k\}_{k\in K})$ over F consists of a morphism $\phi\colon X \to Y$ in $\mathcal{C}$ satisfying that $p_k = q_k \circ \phi$ for all $k \in K$.

2. A cocone $(X, \{i_k\}_{k\in K})$ from F to X consists of an object X of $\mathcal{C}$ together with morphisms $i_k\colon F(k) \to X$ for all $k \in K$ satisfying that

$$i_k = i_l \circ F(\alpha)$$

for all morphisms $\alpha\colon k \to l$ in K.

A morphism of cocones $\phi\colon (X, \{i_k\}_{k\in K}) \to (Y, \{j_k\}_{k\in K})$ over F consists of a morphism $\phi\colon X \to Y$ in $\mathcal{C}$ satisfying that $j_k = \phi \circ i_k$ for all $k \in K$.

Definition 1.3.14. Let $F\colon K \to \mathcal{C}$ be a diagram.

(1) We call a cone $(X, \{p_k\}_{k\in K})$ over F a limit cone if for any cone $(Y, \{q_k\}_{k\in K})$ over F there exists a unique morphism $\beta\colon Y \to X$ in $\mathcal{C}$ satisfying that

$$q_k = p_k \circ \beta$$

for all $k \in K$.
We call an object X of $\mathcal{C}$ a limit of F if there exists a limit cone $(X, \{p_k\}_{k\in K})$ over F, and write $X = \lim(F)$.

(2) We call a cocone $(X, \{i_k\}_{k\in K})$ under F a colimit cocone if for any cocone $(Y, \{j_k\}_{k\in K})$ under F there exists a unique morphism $\beta\colon X \to Y$ in $\mathcal{C}$ satisfying that

$$j_k = \beta \circ i_k$$

for all $k \in K$.
We call an object X of $\mathcal{C}$ a colimit of F if there exists a colimit cocone $(X, \{i_k\}_{k\in K})$ under F, and write $X = \operatorname{colim} F$.

We can rephrase Definition 1.3.14 as follow: a cone $(X, \{p_k\}_{k\in K})$ over F is a limit cone if and only if for any cone $(Y, \{q_k\}_{k\in K})$ over F there exists a unique morphism $\beta\colon (Y, \{q_k\}_{k\in K}) \to (X, \{p_k\}_{k\in K})$ of cones.

Remark 1.3.15. (1) A limit or colimit of a diagram in a category $\mathcal{C}$ does not necessarily exist.

(2) If a diagram admits a (co)limit, the universal property uniquely characterizes the (co)limit up to unique isomorphism, as we show next.

Lemma 1.3.16. *Let $F\colon K \to \mathcal{C}$ be a K-indexed diagram in $\mathcal{C}$. For any two limit diagrams $(X, \{p_k\}_{k\in K}), (X', \{p'_k\}_{k\in K})$ there exists a unique isomorphism $\beta\colon X \simeq X'$ in $\mathcal{C}$ with $p'_k \circ \beta = p_k$ for all $k \in K$ (i.e. a unique isomorphism of cones).*

Proof. Since $(X, \{p_k\}_{k\in K})$ is a limit cone, there exists a unique morphism $\beta\colon X \to X'$ satisfying that $p_k \circ \beta = p'_k$ for all $k \in K$. Similarly, since $(X', \{p'_k\}_{k\in K})$ is a limit cone, there exists a unique morphism $\beta'\colon X' \to X$ satisfying that $p'_k \circ \beta' = p_k$ for all $k \in K$. We thus find that

$$p_k \circ \beta \circ \beta' = p'_k \circ \beta' = p_k$$

for all $k \in K$. Again applying the universal property of $(X, \{p_k\}_{k\in K})$, we find that there exists a unique morphism $\gamma\colon X \to X$ satisfying that $p_k \circ \gamma = p_k$ for all $k \in K$. Therefore, $\gamma = \beta \circ \beta'$. Further, we have $\mathrm{id}_X \circ p_k = p_k$ for all $k \in K$, and thus $\gamma = \mathrm{id}_X$. This shows that β' is a left inverse to β. Switching the roles of X and X', we also find that β' is a right inverse to β, showing that β is an isomorphism. $\square$

Lemma 1.3.17. *Let $F\colon K \to \mathcal{C}$ be a diagram. Then a limit of F exists if and only if a colimit of the opposite diagram $F^{\mathrm{op}}\colon K^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ exists. If a limit of F exists, then there exists an isomorphism in $\mathcal{C}$*

$$\lim F \simeq \operatorname{colim} F^{\mathrm{op}}\,.$$

Proof. A cone $(X, \{p_k\}_{k\in K})$ over F defines a cocone $(X, \{i_k\}_{k\in K})$ under F^{op} with $i_k\colon F^{\mathrm{op}}(k) = F(k) \to X$ given by $p_k \in \mathrm{Hom}_{\mathcal{C}}(X, F(k)) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(F(k), X))$. Inspecting the universal properties, we observe that the former cone is a limit cone if and only if the latter cocone is a colimit cocone. $\square$

Example 1.3.18. Let $\mathcal{C}$ be a category and K a set, considered as a category. A functor $F\colon K \to \mathcal{C}$ amounts to an I-indexed family $(X_k)_{k\in K}$ of objects in $\mathcal{C}$. A limit of F amounts to a product $\prod_{k\in K} X_k$ and a colimit of F amounts to a coproduct $\coprod_{k\in K} X_k$.

Definition 1.3.19. Let K be the category arising from the following poset:

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 1' & \longrightarrow & 2 \end{array}$$

Let $\mathcal{C}$ be a category.

(1) The limit of a functor $F\colon K \to \mathcal{C}$ is called the pullback of F.

(2) The colimit of a functor $F\colon K^{\mathrm{op}} \to \mathcal{C}$ is called the pushout of F.

Let K be as in Definition 1.3.19 and $F\colon K \to \mathcal{C}$ a functor with pullback $F(1) \times_{F(2)} F(1')$. Then the limit cone amounts to a commutative square in $\mathcal{C}$:

$$\begin{tikzcd} F(1) \times_{F(2)} F(1') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(1) \arrow[d] \\ F(1') \arrow[r] & F(2) \end{tikzcd}$$

We indicate that this is a *pullback square* by an angle as above. Similarly, for a functor $G\colon K^{\mathrm{op}} \to \mathcal{C}$ with pushout $G(1) \amalg_{G(2)} G(1')$, the colimit cocone also amounts to a commutative square in $\mathcal{C}$, which we indicate as follows:

$$\begin{tikzcd} G(2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G(1) \arrow[d] \\ G(1') \arrow[r] & G(1) \amalg_{G(2)} G(1') \end{tikzcd}$$

Example 1.3.20. Let $\mathcal{C}$ be a pointed category and $f\colon x \to y$. Then the kernel of f fits into a pullback square

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & x \arrow[d, "f"] \\ 0 \arrow[r] & y \end{tikzcd}$$

and the cokernel of f fits into a pushout square as follows:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & y \arrow[d] \\ 0 \arrow[r] & \mathrm{coker}(f) \end{tikzcd}$$

A category K is called strictly finite if it has finitely many objects and for all objects $k, l \in K$, the set $\mathrm{Hom}_K(k,l)$ is finite. The limit of a diagram $F\colon K \to \mathcal{C}$ indexed by a strictly finite category K is called a finite limit in $\mathcal{C}$. Similarly, a small limit in $\mathcal{C}$ is the limit of a diagram $F\colon K \to \mathcal{C}$ indexed by a small category K.

Lemma 1.3.21.

(1) A category admits small limits if and only if it admits small products and pullbacks.

(2) A category admits strictly finite limits if and only if it admits finite products and pullbacks.

Proof. We begin with the proof of part (1). Let $\mathcal{C}$ be a category. If $\mathcal{C}$ admits small limits, then it clearly admits small products and pullbacks. Conversely, assume that $\mathcal{C}$ admits small products and pullbacks. Let K be a small category and $F\colon K \to \mathcal{C}$ a diagram. By assumption, the $\mathrm{ob}(K)$-indexed family of objects $(F(k))_{k\in\mathrm{ob}(K)}$ admits a product $\prod_{k\in\mathrm{ob}(K)} F(k) \in \mathcal{C}$, with limit cone denoted $(\prod_{k\in\mathrm{ob}(K)} F(k), (p_k)_{k\in K})$. Let $\mathrm{Arr}(K) \coloneqq \amalg_{(k,l)\in\mathrm{ob}(K)\times\mathrm{ob}(K)} \mathrm{Hom}_K(k,l)$ be the set of all morphisms in K. Then the family $(F(l))_{\alpha\colon k\to l\in\mathrm{Arr}(K)}$ similarly admits a product $\prod_{\alpha\colon k\to l\in\mathrm{Arr}(K)} F(l) \in \mathcal{C}$ with limit cone denoted $(\prod_{\alpha\colon k\to l\in\mathrm{Arr}(K)} F(l), (p_\beta)_{\beta\in\mathrm{Arr}(K)})$. We define two morphisms in $\mathcal{C}$

$$s, t\colon \prod_{k\in\mathrm{ob}(K)} F(k) \longrightarrow \prod_{\alpha\colon k\to l\in\mathrm{Arr}(K)} F(l)$$

using the universal property of the product over $\mathrm{Arr}(K)$ by requiring their composites with

$$p_\beta\colon \prod_{\alpha\colon k\to l\in \mathrm{Arr}(K)} F(l)\to F(l')$$

for $\beta\colon k'\to l'$ to be given by $F(\beta)\circ p_{k'}\colon \prod_{k\in \mathrm{ob}(K)} F(k)\to F(l')$ for s and by $p_{l'}\colon \prod_{k\in\mathrm{ob}(K)} F(k)\to F(l')$ for t. In equations, we thus require $p_\beta\circ s = F(\beta)\circ p_{k'}$ and $p_\beta\circ t = p_{l'}$.

We claim that the pullback P of the diagram

$$\begin{tikzcd}
P \arrow[r, "\gamma"] \arrow[d, "\gamma'"] \arrow[dr, phantom, "\lrcorner"] & \prod_{k\in\mathrm{ob}(K)} F(k) \arrow[d, "{(\mathrm{id},s)}"] \\
\prod_{k\in\mathrm{ob}(K)} F(k) \arrow[r, "{(\mathrm{id},t)}"] & \left(\prod_{k\in\mathrm{ob}(K)} F(k)\right)\times\left(\prod_{\alpha\colon k\to l\in\mathrm{Arr}(K)} F(l)\right)
\end{tikzcd} \tag{2}$$

is a limit of F.

By composing the above diagram with the projection

$$\left(\prod_{k\in\mathrm{ob}(K)} F(k)\right)\times\left(\prod_{\alpha\colon k\to l\in\mathrm{Arr}(K)} F(l)\right)\to \prod_{k\in\mathrm{ob}(K)} F(k)\,,$$

we see $\gamma=\gamma'$. We next construct a cone from P to F. For $k\in K$, let $q_k = p_k\circ\gamma\colon P\to\prod_{l\in\mathrm{ob}(K)} F(l)\to F(k)$. We claim that $(P,(q_k)_{k\in K})$ is a cone over F. For any morphism $\beta\colon k\to l$ in K, we have

$$\begin{aligned}
F(\beta)\circ q_k &= F(\beta)\circ p_k\circ\gamma\\
&= p_\beta\circ s\circ\gamma\\
&= p_\beta\circ t\circ\gamma\\
&= p_l\circ\gamma\\
&= q_l\,,
\end{aligned}$$

where the third equality follow from $s\circ\gamma = t\circ\gamma$ by the commutativity of (2), as desired for a cone over F.

Let $(Z,(r_k)_{k\in K})$ be a different cone over F. We define $\delta\colon Z\to\prod_{k\in\mathrm{ob}(K)} F(k)$ by requiring $p_k\circ\delta = r_k$ for all $k\in K$. A small computation as above shows that there is a cone as follows:

$$\begin{tikzcd}
Z \arrow[r, "\delta"] \arrow[d, "\delta"'] & \prod_{k\in\mathrm{ob}(K)} F(k) \arrow[d, "{(\mathrm{id},s)}"] \\
\prod_{k\in\mathrm{ob}(K)} F(k) \arrow[r, "{(\mathrm{id},t)}"] & \left(\prod_{k\in\mathrm{ob}(K)} F(k)\right)\times\left(\prod_{\alpha\colon k\to l\in\mathrm{Arr}(K)} F(l)\right)
\end{tikzcd}$$

By the universal property of the pullback P, there exists a unique morphism $\phi\colon Z\to P$ in $\mathcal{C}$ satisfying that $\gamma\circ\phi=\delta$. Using the universal property of the product over ob(K), we see that the equality $\gamma \circ \phi = \delta$ holds if and only if $q_k \circ \phi = p_k \circ \gamma \circ \phi = p_k \circ \delta = r_k$ holds for all $k \in \mathrm{ob}(K)$. We thus see that ϕ is the unique morphism satisfying that $q_k \circ \phi = r_k$ for all $k \in K$, showing that $(P, (q_k)_{k \in \mathrm{K}})$ is indeed a limit cone over F.

For part (2), we note that if K is strictly finite, then the sets ob(K) and Arr(K) are finite, and the assertion thus follows from the above proof. $\square$

1.3.4 The interchange of limits

The next goal is to show that in any category limits commute with limits.

Construction 1.3.22. Let $F(\text{-},\text{-})\colon K \times L \to \mathcal{C}$ be a functor such that for every $l \in L$, the functor $F(\text{-}, l)\colon K \to \mathcal{C}$ admits a limit $\lim_K F(\text{-}, l) \in \mathcal{C}$. We construct a canonical functor

$$\lim_{k \in K} F(k, \text{-})\colon L \to \mathcal{C}\,,$$

given on objects by $l \mapsto \lim_K F(\text{-}, l)$.

Given a morphism $\alpha\colon l \to l'$ in L, we must define the morphism $\lim_{k \in K} F(k, \alpha)$. Let $(\lim_{k \in K} F(k, l), (p_k^l)_{k \in K})$ be the limit cone over $F(\text{-}, l)$. We then obtain the cone $(\lim_{k \in K} F(k, l), (F(k, \alpha) \circ p_k^l)_{k \in K})$ over $F(\text{-}, l')$. From the universal property of the limit of $F(\text{-}, l')$, we obtain the morphism

$$\phi_\alpha\colon \lim_{k \in K} F(k, \alpha)\colon \lim_{k \in K} F(k, l) \to \lim_{k \in K} F(k, l')\,.$$

It remains to check that this assignment defines a functor: clearly $\phi_{\mathrm{id}_l} = \mathrm{id}_{\lim_{k \in K} F(k,l)}$. Given two morphisms $\alpha\colon l \to l'$ and $\beta\colon l' \to l''$, we must further show that $\phi_\beta \circ \phi_\alpha = \phi_{\beta\circ\alpha}$. By the universal property of the limit $\lim_{k \in K} F(k, l'')$, the morphism $\phi_\beta \circ \phi_\alpha$ is determined by its composition with $p_k^{l''}\colon \lim_{k' \in K} F(k', l'') \to F(k, l'')$. We then have

$$p_k^{l''} \circ \phi_\beta \circ \phi_\alpha = F(k, \beta) \circ p_k^{l'} \circ \phi(\alpha) = F(k, \beta) \circ F(k, \alpha) \circ p_k^l = F(k, \beta \circ \alpha) \circ p_k^l = p_k^l \circ \phi_{\beta\circ\alpha}\,,$$

as desired.

Proposition 1.3.23. *Let $F(\text{-},\text{-})\colon K \times L \to \mathcal{C}$ be a functor such that for every $l \in L$, the functor $F(\text{-}, l)\colon K \to \mathcal{C}$ admits a limit $\lim_K F(\text{-}, l) \in \mathcal{C}$. Then $F(\text{-},\text{-})$ admits a limit if and only if the functor $\lim_{k \in K} F(k, \text{-})\colon L \to \mathcal{C}$ from Construction 1.3.22 admits a limit. Furthermore, in this case there exists an isomorphism*

$$\lim_{K \times L} F(\text{-},\text{-}) \simeq \lim_L \lim_{k \in K} F(k, \text{-}) \in \mathcal{C}\,.$$

Proof. We begin by showing that cones over $\lim_{k \in K} F(k, \text{-})\colon L \to \mathcal{C}$ are in bijection with cones over $F(\text{-},\text{-})\colon K \times L \to \mathcal{C}$.

For $(k, l) \in K \times L$, we denote by $p_k^l\colon \lim_{k' \in K} F(k', l) \to F(k, l)$ the projection map from the limit cone of $F(\text{-}, l)$.

Consider a cone $(X, (q_l)_{l \in L})$ over $\lim_{k \in K} F(k, \text{-})$. For $(k, l) \in K \times L$, we denote by $r_{k,l}$ the composite

$$r_{k,l}\colon X \xrightarrow{q_l} \lim_{k \in K} F(k, l) \xrightarrow{p_k^l} F(k, l)\,.$$

We claim that $(X, (r_{k,l})_{(k,l)\in K\times L})$ defines a cone over F. Indeed, for $(\alpha, \beta)\colon (k, l) \to (k', l')$ in $K \times L$, we have

$$\begin{aligned} F(\alpha,\beta)\circ r_{k,l} &= F(\mathrm{id}_{k'},\beta)\circ F(\alpha,\mathrm{id}_l)\circ p^l_k\circ q_l \\ &= F(\mathrm{id}_{k'},\beta)\circ p^l_{k'}\circ q_l \\ &= p^{l'}_{k'}\circ q_{l'} \\ &= r_{k',l'}\,, \end{aligned}$$

where the second equality uses that $\alpha \circ p^l_k = p^l_{k'}$, and the third equality uses the commutativity of the following square:

$$\begin{tikzcd} \lim_{k\in K} F(k,l) \arrow[r, "p^l_{k'}"] \arrow[d, "\lim_{k\in K} F(k,\beta)"'] & F(k',l) \arrow[d, "F(k',\beta)"] \\ \lim_{k\in K} F(k,l') \arrow[r, "p^{l'}_{k'}"] & F(k',l') \end{tikzcd}$$

Conversely, suppose we are given a cone $(Y, (r_{k,l})_{(k,l)\in K\times L})$ over F. We fix $l' \in L$ and restrict to the cone $(Y, (r_{k,l'})_{k\in K})$ over $F(\text{-}, l')$. By the universal property of the limit, we obtain a morphism $q_{l'}\colon Y \to \lim_{k\in K} F(k, l')$. We verify that $(Y, (q_l)_{l\in L})$ forms a cone over $\lim_{k\in K} F(k, \text{-})$: given a morphism $\beta\colon l \to l'$ in L, we must show that $\lim_{k\in K} F(k, \beta) \circ q_l = q_{l'} \colon Y \to \lim_{k\in K} F(k, l')$. This assertion is equivalent to the statement that $p^{l'}_{k'} \circ \lim_{k\in K} F(k, \beta) \circ q_l = p^{l'}_{k'} \circ q_{l'}$ for each map $p_{k\mathfrak{t}}(l')\colon \lim_{k\in K} F(k, l') \to F(k', l')$, $k' \in K$, from the limit cone. We have

$$p^{l'}_{k'}\circ \lim_{k\in K} F(k,\beta)\circ q_l = F(k',\beta)\circ p^l_{k'}\circ q_l = F(k',\beta)\circ r_{k',l} = r_{k',l'} = p^{l'}_{k'}\circ q_{l'}\,.$$

One readily checks that these assignment between cones are inverse to each other.

It remains to check that the assignments are functorial with respect to morphisms of cones, and hence identify the universal properties of the two limits. Let $\psi\colon (X, (q_l)_{l\in L}) \to (Y, (q'_l)_{l\in L})$ be a morphism between cones over $\lim_{k\in K} F(k, \text{-})$. Clearly, after composing the cones with $p_k(l)$, the morphism ψ defines a morphism between the cones $(X, (r_{k,l} = p^l_k \circ q_l)_{(k,l)\in K\times L})$ and $(Y, (r'_{k,l} = p^l_k \circ q'_l)_{(k,l)\in K\times L})$ over $F(\text{-}, \text{-})$.

Conversely, consider a morphism of cones $\psi\colon (X, (r_{k,l})_{(k,l)\in K\times L}) \to (Y, (r'_{k,l})_{(k,l)\in K\times L})$ over F. Let $(X, (q_l)_{l\in L}), (Y, (q'_l)_{l\in L})$ be the two cones over $\lim_{k\in K} F(k, \text{-})$ obtained as above. Then $p^l_k \circ q'_l \circ \psi = r'_{k,l} \circ \psi = r_{k,l} = p_k(l) \circ q_l$ for all $(k, l) \in K \times L$ and thus $q'_l \circ \psi = q_l$ for all $l \in L$ by the universal property of the limit $\lim_{k\in K} F(k, l)$. We thus see that ψ defines the desired morphism of cones $\psi\colon (X, (q_l)_{l\in L}) \to (Y, (q'_l)_{l\in L})$. □

Corollary 1.3.24. *Let $F\colon K \times L \to \mathcal{C}$ be a diagram with K, L small categories. If F admits a limit and the limits of $F(k, \text{-}), F(\text{-}, l)$ exist for all $k \in K, l \in L$, then*

$$\lim_K \lim_{l\in L} F(\text{-}, l) \simeq \lim_L \lim_{k\in K} F(k, \text{-})\,.$$

We note that in general, colimits do not commute with limits:

Example 1.3.25. Consider the group $G = \mathbb{Z}/2\mathbb{Z}$, with the action on $\mathbb{Z}$ determined by $1.x = -x \in \mathbb{Z}$. Consider the following pullback and pushout square in Ab:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{Z} \arrow[d, "x \mapsto [x]"] \\ 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

Let BG be the category with a single object $*$ and $\mathrm{Hom}_{BG}(*, *) = G$. Let $Q = \{0 \to 1, 1' \to 2\}$ be a commutative square (considered as a poset category). The above square extends to a functor

$$BG \times Q \to \mathrm{Ab}$$

encoding the G-actions and the fact the above morphisms of abelian groups are equivariant with respect to the G-actions (meaning they are G-homomorphisms in the terminology of Einführung in die Algebra). Note that the G-action on $\mathbb{Z}/2\mathbb{Z}$ is given by $1.[x] = [1.x] = [-x] = [x]$, and thus trivial. Passing to the limit over BG amounts to passing to the G-fixed points, which yields the following commutative square in Ab:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

This square is not pushout, but still pullback, in line with Proposition 1.3.23, which guarantees that limits (such as pullbacks) commute with limits (such as passing to G-fixed points $= \lim_{BG}$).

1.4 Additive and abelian categories

1.4.1 Additive categories

Definition 1.4.1. An Ab-enriched category consists of a category $\mathcal{C}$ together with the structure of an abelian group on the Hom set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ for all objects x, y of $\mathcal{C}$, such that for all objects x, y, z of $\mathcal{C}$ the composition morphism

$$- \circ -\colon\ \mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

is bilinear.

We also refer to Ab-enriched categories as Ab-categories for brevity.

Definition 1.4.2. An Ab-category $\mathcal{C}$ is called additive if it admits finite products.

We will also abusively call a category $\mathcal{C}$ additive, if it admits the structure of an Ab-category which is additive.

Lemma 1.4.3. *Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is pointed.*

Proof. Since $\mathcal{C}$ admits finite products, $\mathcal{C}$ admits a terminal object $* \in \mathcal{C}$. Then $\mathrm{id}_* \in \mathrm{Hom}_{\mathcal{C}}(*, *) = \{\mathrm{id}_*\}$ must be the neutral element in the abelian group. Let $x \in \mathcal{C}$. We show that $\mathrm{Hom}_{\mathcal{C}}(*, x)$ contains a unique element. Firstly, we note that the abelian group $\mathrm{Hom}_{\mathcal{C}}(*, x)$ contains a neutral element $0_{*\to x}$. Let $f\colon * \to x$ be any morphism. Then

$$f = f \circ \mathrm{id}_* = f \circ 0_{*\to *} = 0_{*\to x}\,,$$

since composition is bilinear[1]. This shows that $\mathrm{Hom}_{\mathcal{C}}(*, x) = \{0_{*\to x}\}$, showing that $*$ is also initial in $\mathcal{C}$. $\square$

Given a morphism $f\colon x \to y$ in an additive category $\mathcal{C}$, we write $f = 0$, as in Section 1.3.2, if f factors as $x \to 0 \to y$. This notation is compatible with the Ab-enrichment:

Lemma 1.4.4. *Let $\mathcal{C}$ be an additive category and $f\colon x \to y$ a morphism in $\mathcal{C}$. Then $f = 0$ if and only if $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ is the neutral element of the abelian group.*

Proof. Since composition is bilinear, the neutral element $0_{x\to y} \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ is the composite of the neutral elements $0_{x\to 0} \in \mathrm{Hom}_{\mathcal{C}}(x, 0)$ and $0_{0\to y} \in \mathrm{Hom}_{\mathcal{C}}(0, y)$. Note that $\mathrm{Hom}_{\mathcal{C}}(x, 0) = \{0_{x\to 0}\}$ as a set, since 0 is terminal, and similarly $\mathrm{Hom}_{\mathcal{C}}(0, y) = \{0_{0\to y}\}$. Thus, f factors as $x \to 0 \to y$ if and only if $f = 0_{x\to y} \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. $\square$

Lemma 1.4.5. *Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ admits finite coproducts. Futhermore, for any finite family $(x_i)_{i\in I}$ of objects in $\mathcal{C}$, there exists an isomorphism*

$$\prod_{i\in I} x_i \simeq \coprod_{i\in I} x_i\,.$$

Proof. If $I = \emptyset$, then $\prod_{i\in I} x_i \simeq \coprod_{i\in I} x_i = 0 \in \mathcal{C}$, since $\mathcal{C}$ is pointed. If $I \simeq \{1\}$ contains a unique element, then $\prod_{i\in I} x_i \simeq x_1 \simeq \coprod_{i\in I} x_i$. We prove below the assertion in the case $I \simeq \{1, 2\}$. In the remaining cases where $I \simeq \{1, \dots, n\}$ with $n > 2$, we have

$$\begin{aligned}\prod_{i\in I} x_i &\simeq ((((x_1 \times x_2) \times x_3) \times \dots) \times x_n)\\ &\simeq ((((x_1 \amalg x_2) \amalg x_3) \amalg \dots) \amalg x_n)\\ &\simeq \coprod_{i\in I} x_i\,.\end{aligned}$$

Assume that $I = \{1, 2\}$. Let $x_1 \times x_2$ be the product of x_1, x_2 with projection maps $p_1\colon x_1 \times x_2 \to x_1$ and $p_2\colon x_1 \times x_2 \to x_2$. Using the universal property of the product, we define two morphism $i_j\colon x_j \to x_1 \times x_2$, with $j = 1, 2$, by requiring

$$p_k \circ i_j = \begin{cases} \mathrm{id}_{x_j} & j = k\,,\\ 0 & j \neq k\,.\end{cases}$$

[1]Bilinearity is used here in the last step follows: we have $f \circ 0 = f \circ (0 - 0) = f \circ 0 - f \circ 0 = 0$.

We next show that $(x_1 \times x_2, (i_j)_{j=1,2})$ is a colimit cocone, thus exhibiting $x_1 \times x_2$ as the coproduct.

We have $\mathrm{id}_{x_1\times x_2} = i_1 \circ p_1 + i_2 \circ p_2$: Using the universal property of the product, it suffices to note that

$$p_1 \circ (i_1 \circ p_1 + i_2 \circ p_2) = \mathrm{id}_{x_1} \circ p_1 + 0 \circ p_2 = p_1$$

and

$$p_2 \circ (i_1 \circ p_1 + i_2 \circ p_2) = 0 \circ p_1 + \mathrm{id}_{x_2} \circ p_2 = p_2 \,.$$

Let $y \in \mathcal{C}$ be an object with two morphisms $i'_j\colon x_j \to y$, $j = 1, 2$. We must show that there is a unique morphism $\phi\colon x_1 \times x_2 \to y$ of cocones, i.e. satisfing $i'_j = \phi \circ i_j$ for $j = 1, 2$. Given any such morphism ϕ, we have

$$\phi = \phi \circ \mathrm{id}_{x_1\times x_2} = \phi \circ ((i_1 \circ p_1) + (i_2 \circ p_2)) = i'_1 \circ p_1 + i'_2 \circ p_2 \,.$$

This morphism ϕ indeed defines the desired, then unique, morphism of cocones: We have

$$\phi \circ i_1 = i'_1 \circ (p_1 \circ i_1 + i'_2 \circ p_2 \circ i_1) = i'_1 \circ \mathrm{id}_{x_1} + i'_2 \circ 0 = i'_1$$

and similarly $\phi \circ i_2 = i'_2$. □

Remark 1.4.6. Let $\mathcal{C}$ be an additive category and $(x_i)_{i\in I}$ a finite family of objects in $\mathcal{C}$. To emphasize that the product and coproduct of $(x_i)_{i\in I}$, coincide, we write it as

$$\bigoplus_{i\in I} x_i = \coprod_{i\in I} x_i \simeq \prod_{i\in I} x_i \,,$$

referring to it as the direct sum of the family $(x_i)_{i\in I}$.

Definition 1.4.7. Let $\mathcal{C}, \mathcal{D}$ be additive categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called additive if for each pair of objects $x, y \in \mathcal{C}$, the morphism

$$F\colon\ \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$

is a morphism of abelian groups.

Exercise 1.4.8. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Then F preserves finite products and coproducts.

Remark 1.4.9. Let $n \geq 1$. In an additive category, we can write a morphism

$$f = (f_i)_{i=1,\ldots,n}\colon A \to \bigoplus_{i=1}^{n} B_i$$

as a tuple of morphisms $f_i\colon A \to B_i$, using the universal property of the product. Similarly, we can write a morphism

$$g = (g_i)_{i=1,\ldots,n}\colon \bigoplus_{i=1}^{n} B_i \to C$$

as a tuple with $g_i\colon B_i \to C$, using the universal property of the coproduct. Further, the composition is given by sum

$$g \circ f = \sum_{i=1}^{n} g_i \circ f_i \,.$$

Lemma 1.4.10. *Let $\mathcal{C}$ be an additive category. Then a square in $\mathcal{C}$*

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "f'"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \tag{3}$$

is a pullback square if and only if the folded square in $\mathcal{C}$

$$\begin{tikzcd} P \arrow[r, "{(g',f')}"] \arrow[d] & A \oplus B \arrow[d, "{(f,-g)}"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{4}$$

is a pullback square.
Similarly, the square (3) is pushout if and only if the square (4) is pushout.

Proof. We first show that there exists a bijection between cones over the two diagrams:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \qquad \begin{tikzcd} & A \oplus B \arrow[d, "{(f,-g)}"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{5}$$

A morphism $h\colon Z \to A \oplus B$ amounts via the universal property of the product to a pair of morphisms $h_1\colon Z \to A$ and $h_2\colon Z \to B$. There is a unique morphism $P \to 0$ since 0 is terminal. Further the condition $(f, -g) \circ h = 0$ is equivalent to

$$0 = (f, -g) \circ \mathrm{id}_{A \oplus B} \circ h = (f, -g) \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ h = f \circ h_1 - g \circ h_2 \,,$$

meaning that $f \circ h_1 = g \circ h_2$. Thus, both the data in the two diagrams, as well as their commutativity,

$$\begin{tikzcd} Z \arrow[r, "h_1"] \arrow[d, "h_2"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "{h=(h_1,h_2)}"] \arrow[d] & A \oplus B \arrow[d, "{(f,-g)}"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{6}$$

are equivalent, showing the bijection between cones.

To identify the morphisms of cones over the two diagrams, it remains to note that a diagram

$$\begin{tikzcd} Z \arrow[dr, "\phi"] \arrow[drr, bend left, "h_1"] \arrow[ddr, bend right, "h_2"'] & & \\ & P \arrow[r, "g'"] \arrow[d, "f'"] & A \arrow[d, "f"] \\ & B \arrow[r, "g"] & C \end{tikzcd}$$

commutes if and only if the diagram

$$
\begin{tikzcd}
Z \arrow[dr, "\phi"] \arrow[drr, bend left, "h=(h_1,h_2)"] \arrow[ddr, bend right] & & \\
& P \arrow[r, "(g',f')"] \arrow[d] & A \oplus B \arrow[d, "(f,-g)"] \\
& 0 \arrow[r] & C
\end{tikzcd}
$$

commutes. Indeed, we have $(g', f') \circ \phi = (h_1, h_2)$ if and only if $g' \circ \phi = h_1$ and $f' \circ \phi = h_2$. $\square$

1.4.2 Abelian categories

Definition 1.4.11. Let $f \colon x \to y$ be a morphism in a category $\mathcal{C}$.

(1) Then f is called a monomorphism if for all morphisms $g_1, g_2 \colon w \to x$ in $\mathcal{C}$ with $f \circ g_1 = f \circ g_2$, it also holds that $g_1 = g_2$.

(2) Then f is called an epimorphism if for all morphisms $g_1, g_2 \colon y \to z$ in $\mathcal{C}$ with $g_1 \circ f = g_2 \circ f$, it also holds that $g_1 = g_2$.

Remark 1.4.12. If $\mathcal{C}$ is an additive category and f a morphism in $\mathcal{C}$, then f is a monomorphism if and only if $f \circ g = 0$ implies that $g = 0$. Indeed, assuming the latter, if $f \circ g_1 = f \circ g_2$, then $f \circ (g_1 - g_2) = 0$, which implies $g_1 - g_2 = 0$, which is equivalent to $g_1 = g_2$.

Similarly, f is an epimorphism if and only if $g \circ f = 0$ implies $g = 0$.

Definition 1.4.13. An additive category $\mathcal{C}$ is called abelian if

- the category $\mathcal{C}$ admits kernels and cokernels,
- for every commutative square in $\mathcal{C}$ of the form

$$
\begin{tikzcd}
x \arrow[r, "i"] \arrow[d] & y \arrow[d, "p"] \\
0 \arrow[r] & z
\end{tikzcd}
\tag{7}
$$

  the following are two statements are equivalent:

  i) The morphism i is a monomorphism and the square is a pushout square.

  ii) The morphism p is an epimorphism and the square is a pullback square.

Lemma 1.4.14. *Let $f \colon x \to y$ be a morphism in an additive category $\mathcal{C}$.*

(1) If f admits a kernel, then the morphism $i \colon \ker(f) \to x$ is a monomorphism.

(2) If f admits a cokernel, then the morphism $p\colon y \to \operatorname{coker}(f)$ is an epimorphism.

Proof. We only prove part (1), part (2) follows from the same argument applied to $\mathcal{C}^{\mathrm{op}}$. Let $g\colon z \to \ker(f)$ be a morphism with $i \circ g = 0$. By the universal property of the kernel, the two morphisms $g, 0\colon z \to \ker(f)$ agree if and only if $i \circ g = i \circ 0$, which indeed holds since both morphisms are zero. Thus $g = 0$, as desired. □

Remark 1.4.15. By Lemma 1.4.14, we find that the second condition in Definition 1.4.13 is equivalent to the a priori weaker assumption that if i is a monomorphism and (7) pushout, then (7) is also pullback, and that if p is an epimorphism and (7) pullback, then (7) is also pushout.

The prototypical example of an abelian category is the category Ab of abelian groups. Categories of left modules are also abelian:

Proposition 1.4.16. *Let R be a ring. Then LMod_R is an abelian category.*

Proof. By Remark 1.1.7, we can equip the morphisms sets in LMod_R with the structures of abelian groups. It is straightforward to check that the composition of morphisms in LMod_R is bilinear with respect to these group structures. By Example 1.3.5 the category LMod_R admits finite products. Thus LMod_R is additive.

By Proposition 1.3.10, any morphism in the category LMod_R admits a kernel and a cokernel.

Monomorphisms in LMod_R are given by inclusions of R-modules and epimorphisms in LMod_R are given by surjective morphisms between R-modules. Let $i\colon M' \subset M$ be an inclusion of R-modules. We observe that the kernel of the morphism $M \to M/M'$ is given by M'. Let $p\colon M \to N$ be a surjective morphism of R-modules. Then $N \simeq M/\ker(p)$ by the first isomorphism theorem. This shows that LMod_R is abelian. □

Definition 1.4.17. Let $\mathcal{C}$ be an abelian category and $f\colon x \to y$ a morphism in $\mathcal{C}$. The image of f is defined as

$$\mathrm{Im}(f) \coloneqq \ker(y \to \operatorname{coker}(f)) .$$

Remark 1.4.18. (1) Any morphism $f\colon x \to y$ in an abelian category factors as

$$x \xrightarrow{p} \mathrm{Im}(f) \xrightarrow{i} y$$

with p an epimorphism and i a monomorphism. This follows from the observation that the composite morphism $x \xrightarrow{f} y \to \operatorname{coker}(f)$ vanishes and thus factors through the kernel of $y \to \operatorname{coker}(f)$.

(2) Let $f\colon M \to N$ be a morphism between left R-modules. Then the image of f can be identified with the submodule of N with elements

$$\mathrm{Im}(f) = \{f(m) | m \in M\} \subset N .$$

Proposition 1.4.19. *Let $\mathcal{C}$ be an abelian category. Then $\mathcal{C}$ admits strictly finite limits and colimits.*

Proof. By Lemma 1.3.21, it suffices to show that $\mathcal{C}$ admits finite products and coproducts, as well as pullbacks and pushouts. The former follows from the assumption that $\mathcal{C}$ is additive. The latter follows from Lemma 1.4.10 and the fact that $\mathcal{C}$ admits kernels and cokernels. $\square$

We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves limits if for each limit cone $(X, (p_z)_{z\in Z})$ in $\mathcal{C}$ over a functor $g\colon Z \to \mathcal{C}$, the cone $(F(X), (F(p_z))_{z\in Z})$ in $\mathcal{D}$ over $F \circ g\colon Z \to \mathcal{D}$ is also a limit cone.

Definition 1.4.20. Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $F\colon \mathcal{C} \to \mathcal{D}$ a functor.

(1) The functor F is called left exact if it is additive and preserves kernels.

(2) The functor F is called right exact if it is additive and preserves cokernels.

(3) The functor F is called exact if it is both left exact and right exact.

Lemma 1.4.21. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $F\colon \mathcal{C} \to \mathcal{D}$ an additive functor.*

(1) The functor F is left exact if and only if F preserves strictly finite limits.

(2) The functor F is right exact if and only if F preserves strictly finite colimits.

(3) The functor F is exact if and only if F preserves strictly finite limits and finite colimits.

Proof. Part (3) follows directly from parts (1) and (2). Part (2) follows from part (1) applied to $\mathcal{C}^{\mathrm{op}}$. We turn to the proof of part (1). We first show that F is left exact if and only if F preserves pullbacks. Clearly, the latter implies the former. Conversely, consider a pullback square

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \tag{8}$$

in $\mathcal{C}$. Consider the folded pullback square

$$\begin{tikzcd} P \arrow[r, "{(g',f')}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \oplus B \arrow[d, "{(f,-g)}"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{9}$$

see also Lemma 1.4.10. Applying F to (8) and folding yields the same square as applying F to (9) (this uses that F is an additive functor and thus preserves direct sums). Since F preserves kernels, applying F to (9) yields a pullback square.

Unfolding it and applying Lemma 1.4.10, we find that F applied to (8) is also pullback, showing that F preserves pullbacks.

We next show via a similar argument that if F preserves pullbacks, then F preserves strictly finite limits, using Lemma 1.3.21 in place of Lemma 1.4.10. We fix a strictly finite category Z and a functor $G\colon Z \to \mathcal{C}$. Then G admits a limit if and only if there exists a pullback square as in (2), and in this case the limit of G is isomorphic to the pullback P in (2). Applying F to the pullback square in (2) yields by assumption a pullback square. Again arguing as in the proof of Lemma 1.3.21, we find that the limit of $F \circ G$ is equivalent to $F(P)$ and that the cone from $F(P)$ to $F \circ G$ is indeed a limit cone. $\square$

Example 1.4.22 (Hom is left exact)**.** Given an abelian category $\mathcal{C}$ and $X \in \mathcal{C}$, the functors

$$\mathrm{Hom}_{\mathcal{C}}(X, \text{-})\colon \mathcal{C} \to \mathrm{Ab}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(\text{-}, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$$

are left exact (this follows from a similar argument as on exercise sheet 1, by using Theorem 1.4.24 below).
This can also be deduced from a general fact in category theory: given any category $\mathcal{D}$ and $Y \in \mathcal{D}$, the functor $\mathrm{Hom}_{\mathcal{D}}(X, \text{-})\colon \mathcal{D} \to \mathrm{Set}$ preserves all limits which exist in $\mathcal{D}$. Note that the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$ preserves limits.

In a general abelian category, the objects do not have element, see for instance any abelian category of sheaves. Theorem 1.4.24 below shows that we can embed any small abelian category into an abelican category whose objects have elements.

Definition 1.4.23. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

(1) The functor F is called full if, for all $x, y \in \mathcal{C}$, the morphism $F\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ is surjective,

(2) The functor F is called faithful if, for all $x, y \in \mathcal{C}$, the morphism $F\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ is injective,

(3) The functor F is called fully faithful if F is full and faithful.

For instance, the inclusion of a full subcategory $\mathcal{C}' \subset \mathcal{C}$ defines a fully faithful functor.

Theorem 1.4.24 (The Freyd-Mitchell embedding theorem)**.** *Let $\mathcal{C}$ be a small abelian category. Then there exists a ring R and a fully faithful exact functor*

$$\mathcal{C} \hookrightarrow \mathrm{LMod}_R\ .$$

1.4.3 A digression on Grothendieck universes

In a few cases, we have been relying on our abelian categories being small. This assumption can be inconvenient in practice, as many abelian categories of interest are not small (such as LMod_R, or also abelian categories of sheaves). In practice, to avoid the assumption on smallness, one employs Grothendieck universes and the Grothendieck universe axiom.

A set U (containing only sets itself) is called a Grothendieck universe if it is closed under the following typical operations:

- if $x \in U$ and $y \in x$, then $y \in U$,
- if $x \in U$, then $\mathcal{P}(x) \in U$, with $\mathcal{P}(x)$ the power set,
- $\emptyset \in U$,
- if $x, y \in U$, then $\{x, y\} \in U$,
- if $\{x_i\}_{i \in I}$ is an I-indexed family of elements of U and $I \in U$, then $\bigcup_{i \in I} x_i \in U$.

Instead of considering small sets, one can talk about U-small sets.

Definition 1.4.25. Let U be a Grothendieck universe. A set I is called U-small if $I \in U$.

Typically, the Grothendieck universe U will be fixed, and we will refer to U-small sets as small sets.

With a fixed choice of universe U, when we write LMod_R, we mean the abelian category whose objects are U-small R-modules, meaning R-module M, whose underlying set satisfies $M \in U$. The category LMod_R is then itself not U-small, but instead $\mathrm{LMod}_R \subset U$ is a subset. To proceed, one imposes the following axiom:

Axiom: *For every set I, there exists a Grothendieck universe U with $I \in U$.*

This ensures that LMod_R is an element of a larger universe U', so that we may treat LMod_R still as small (relative to the larger universe U').

The above universe axiom has various implications going beyond ZFC, such as the existence of inaccessible cardinals, and leads to its own version of set theory, called Tarski–Grothendieck set theory.

1.5 Chain complexes

Definition 1.5.1. Let $\mathcal{C}$ be an additive category.

(1) An (unbounded) chain complex A_* in $\mathcal{C}$ consists of a family $(A_n)_{n \in \mathbb{Z}}$ of objects of $\mathcal{C}$ and a family of morphisms $(d_n \colon A_n \to A_{n-1})_{n \in \mathbb{Z}}$ in $\mathcal{C}$, satisfying that

$$d_{n-1} \circ d_n = 0$$

for all $n \in \mathbb{Z}$.

A chain complex A_* is called bounded if there exist $n, m \geq 0$ such that $A_k \simeq 0$ for all $k \notin [-m, n]$.

(2) An (unbounded) cochain complex A^* in $\mathcal{C}$ consists of a family $(A^n)_{n\in\mathbb{Z}}$ of objects of $\mathcal{C}$ and a family of morphisms $(d^n\colon A^n \to A^{n+1})_{n\in\mathbb{Z}}$ in $\mathcal{C}$, satisfying that

$$d^{n+1} \circ d^n = 0$$

for all $n \in \mathbb{Z}$.

The difference between chain complexes and cochain complexes is cosmetic, but distinguishing them can be helpful in practice. In the literature, there is no agreed upon convention on whether one should use by default chain complexes or cochain complexes.

Definition 1.5.2. Let $\mathcal{C}$ be an abelian category.

(1) Let A_* be a chain complex in $\mathcal{C}$. The n-chains of A_* are defined as $C_n(A_*) = A_n \in \mathcal{C}$. The n-cycles of A_* are defined as $Z_n(A_*) = \ker(d_n) \in \mathcal{C}$. The n-boundaries of A_* are defined as $B_n(A_*) = \mathrm{Im}(d_{n+1}) \in \mathcal{C}$.

(2) Let A^* be a cochain complex in $\mathcal{C}$. The n-cochains of A^* are defined as $C^n(A^*) = A^n \in \mathcal{C}$. The n-cocycles of A^* are defined as $Z^n(A^*) = \ker(d^n) \in \mathcal{C}$. The n-coboundaries of A^* are defined as $B^n(A^*) = \mathrm{Im}(d^{n-1}) \in \mathcal{C}$.

If A_* is chain complex, then there are canonical monomorphisms $B_n(A_*) \to Z_n(A_*) \to C_n(A_*)$.

Definition 1.5.3. Let $\mathcal{C}$ be an abelian category.

(1) For $n \in \mathbb{Z}$, the n-th homology of A_* is defined as

$$H_n(A_*) \coloneqq \mathrm{coker}(\mathrm{Im}(d_{n+1}) \hookrightarrow \ker(d_n)) = \mathrm{coker}(B_n(A_*) \hookrightarrow Z_n(A_*))\,.$$

(2) Let A^* be a cochain complex in $\mathcal{C}$. For $n \in \mathbb{Z}$, the n-th cohomology of A^* is defined as

$$H^n(A^*) = \mathrm{coker}(\mathrm{Im}(d^{n-1}) \hookrightarrow \ker(d^n)) = \mathrm{coker}(B^n(A^*) \hookrightarrow Z^n(A^*))\,.$$

If $f\colon A \hookrightarrow B$ is a monomorphism in an abelian category, we also write B/A for $\mathrm{coker}(f)$. For a chain complex A_*, we thus have $H_n(A_*) = \ker(d_n)/\mathrm{Im}(d_{n+1})$.

Definition 1.5.4. Let A_* be a chain complex. For $n \in \mathbb{Z}$, A_* is called exact in position n if $H_n(A_*) \simeq 0$. Further, A_* is called exact if A_* is exact in all positions $n \in \mathbb{Z}$.

The homology of a chain complex is therefore the measure of its failure to be exact.

Chain maps between chain complexes in abelian categories are defined as in Definition 0.5, i.e. as a collection of maps $f_*\colon A_* \to B_*$ satisfying $d^B_{n-1} \circ f_n = f_{n-1} \circ d^A_n$.

Definition 1.5.5. Let $\mathcal{C}$ be an abelian category. Then the category $\mathrm{Ch}_*(\mathcal{C})$ of chain complexes in $\mathcal{C}$ has as objects unbounded chain complexes in $\mathcal{C}$ and as morphisms chain maps. Composition is the (naive) composition of chain maps.

Exercise 1.5.6. Let $\mathcal{C}$ be an abelian category.

- The category of chain complexes $\mathrm{Ch}_*(\mathcal{C})$ is abelian.
- A chain map $f_*\colon A_* \to B_*$ is a monomorphism if and only if $f_n\colon A_n \to B_n$ is a monomorphism for all $n \in \mathbb{Z}$.
- A chain map $f_*\colon A_* \to B_*$ is an epimorphism if and only if $f_n\colon A_n \to B_n$ is an epimorphism for all $n \in \mathbb{Z}$.

Remark 1.5.7. For every $n \in \mathbb{Z}$, the passage to the n-homology defines an additive functor $H_n(\text{-})\colon \mathrm{Ch}_*(\mathcal{C}) \to \mathcal{C}$.

Definition 1.5.8. A chain map $f_*\colon A_* \to B_*$ is called a quasi-isomorphism if $H_n(f_*)\colon H_n(A_*) \to H_n(B_*)$ is an isomorphism of abelian groups for all $n \in \mathbb{Z}$.

1.5.1 Short exact sequences

Lemma 1.5.9 (The 5-Lemma)**.** *Consider a commutative diagram in a small abelian category* $\mathcal{C}$

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & X_2 \arrow[r, "f_2"] \arrow[d, "\beta"] & X_3 \arrow[r, "f_3"] \arrow[d, "\gamma"] & X_4 \arrow[r, "f_4"] \arrow[d, "\delta"] & X_5 \arrow[d, "\epsilon"] \\ Y_1 \arrow[r, "g_1"] & Y_2 \arrow[r, "g_2"] & Y_3 \arrow[r, "g_3"] & Y_4 \arrow[r, "g_4"] & Y_5 \end{tikzcd} \tag{10}$$

such that the two rows are exact chain complexes. Then if $\alpha, \beta, \delta, \epsilon$ are isomorphisms, so is γ.

Proof. Using Theorem 1.4.24, we may choose an exact fully faithful functor $i\colon \mathcal{C} \to \mathrm{LMod}_R$ and equivalently prove the assertion for the image of the diagram (10) under i. We thus assume in the following that the objects in (10) are R-modules.

We first show that γ is a monomorphism. For this, we can equivalently show that for any $x \in X_3$ the equation $\gamma(x) = 0$ implies $x = 0$. We thus fix $x \in X_3$ with $\gamma(x) = 0 \in Y_3$. Then $\delta \circ f_3(x) = g_3 \circ \gamma(x) = g_3(0) = 0$. Since δ is an isomorphism (an in particular a monomorphism), we find that $f_3(x) = 0$. Since the upper row of (10) is exact, there thus exists an element $x' \in X_2$ with $f_2(x') = x$. We next find that $g_2 \circ \beta(x') = \gamma \circ f_2(x') = \gamma(x) = 0$. Since the lower row in (10) is exact, it follows that there exists $y \in Y_1$ with $g_1(y) = \beta(x')$. The object y admits an inverse image x'' under α. We have

$$\beta \circ f_1(x'') = g_1 \circ \alpha(x'') = g_1(y) = \beta(x')\,.$$

Since β is a monomorphism, this shows that $f_1(x'') = x'$. Using that $0 = f_2 \circ f_1$ since the upper row is a chain complex, we finally obtain $0 = f_2 \circ f_1(x'') = f_2(x') = x$, as desired.

Next, we prove that γ is an epimorphism. For this, we fix an element $y \in Y_3$ and show that it has an inverse image under γ. Since δ is an epimorphism, there exists $x \in X_4$ with $\delta(x) = g_3(y)$. We further have

$$\epsilon \circ f_4(x) = g_4 \circ \delta(x) = g_4 \circ g_3(y) = 0\,.$$

Since ϵ is a monomorphism, this shows that $f_4(x) = 0$. By the exactness of the upper row in (10), there exists $x' \in X_3$ with $f_3(x') = x$. Then

$$g_3(\gamma(x') - y) = \delta \circ f_3(x') - g_3(y) = \delta(x) - g_3(y) = 0\,.$$

Thus, there exists $y' \in Y_2$ with $g_2(y') = \gamma(x') - y$. Using that β is an epimorphism, we can find $x'' \in X_2$ with $\beta(x'') = y'$. We have

$$\gamma(x' - f_2(x'')) = \gamma(x') - g_2 \circ \beta(x'') = \gamma(x') - g_2(y') = \gamma(x') - (\gamma(x') - y) = y\,.$$

Thus $x' - f_2(x'') \in X_3$ is the desired inverse image of y, showing that γ is an epimorphism.

Since γ is both an epimorphism an a monomorphism and $\mathcal{C}$ abelian, we find that γ is an isomorphism. □

The technique used to prove Lemma 1.5.9 is called *diagram chasing*. It is a standard proof technique used in homological algebra. See also here.

Definition 1.5.10. A short exact sequence in an abelian category is an exact chain complex of the form

$$0 \to A \to B \to C \to 0\,.$$

Proposition 1.5.11. *A chain complex* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in an abelian category is exact if and only if the corresponding commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{11}$$

is both pushout and pullback.

Proof. The exactness of the sequence at A amounts to f being a monomorphism. Exactness at C amounts to g being an epimorphism. Finally, exactness at B amounts to $\mathrm{Im}(f) = \ker(g)$.

It follows from the axioms of an abelian cateogry, that the square (11) is pullback and pushout if and only if f is a monomorphism, g is an epimorphism and $A = \ker(g)$. In this case, $A \simeq \mathrm{Im}(f)$ so that the assertion $A = \ker(g)$ is equivalent to $\mathrm{Im}(f) = \ker(g)$. □

Let $\mathcal{C}$ be an abelian category and $X, Y \in \mathcal{C}$ and extension of X by Y consists of a short exact sequence $Y \to E \to X$ in $\mathcal{C}$. We introduce the following equivalence relation on extensions:

Lemma 1.5.12. *Let $\mathcal{C}$ be an abelian category and $X, Y \in \mathcal{C}$. There exists an equivalence relation on the set of extensions $\{Y \to E \to X\}$ of X by Y in $\mathcal{C}$ where*

$$(Y \to E \to X) \quad \sim \quad (Y \to E' \to X)$$

if there exists a commutative diagram of the following form:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d, "\alpha"] & X \arrow[r] \arrow[d, "="] & 0 \arrow[d] \\ 0 \arrow[r] & Y \arrow[r] & E' \arrow[r] & X \arrow[r] & 0 \end{tikzcd} \tag{12}$$

Proof. Reflexivity of the relation is clear (set $\alpha = \mathrm{id}_E$). The symmetry follows from the 5-Lemma: the morphism α in (12) is an isomorphism. The transitivity is then also clear. ☐

Definition 1.5.13. Let $\mathcal{C}$ be a small abelian category and $X, Y \in \mathcal{C}$. We denote by

$$\mathrm{Ext}_{\mathcal{C}}(X, Y) = \{Y \to E \to X \text{ extension in } \mathcal{C}\}/\sim$$

the set of equivalence classes of extensions of X by Y.

Lemma 1.5.14. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a short exact sequence in an abelian category. The following are equivalent:*

1) g admits a section s, meaning a morphism $s\colon C \to B$ satisfying that $g \circ s = \mathrm{id}_{\mathcal{C}}$.

2) f admits a retract r, meaning a morphism $r\colon B \to A$ satisfying that $r \circ f = \mathrm{id}_A$.

3) The extension is equivalent, in the sense of Lemma 1.5.12, to the trivial extension

$$A \xrightarrow{(\mathrm{id}_A, 0)} A \oplus C \xrightarrow{(0, \mathrm{id}_C)} C\,.$$

If the above conditions are satisfied, we refer to the short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ as split.

Proof. Clearly, 3) implies 1) and 2). We next show that 1) implies 3). A similar argument shows that 2) implies 3).

Let s be the section of g, satisfying $g \circ s = \mathrm{id}_C$. Then the following commutative diagram shows 3):

$$\begin{tikzcd} A \arrow[r, "(\mathrm{id}_A, 0)"] \arrow[d, "="'] & A \oplus C \arrow[r, "(0, \mathrm{id}_C)"] \arrow[d, "(f, s)"] & C \arrow[d, "="] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

☐

Remark 1.5.15. Any additive functor between abelian categories preserves direct sums as well as the associated inclusions and projections (since these appear in the corresponding limit and colimit cones), and thus maps split short exact sequences to split short exact sequences.

The notion of a split short exact sequence generalizes to split chain complexes as follows:

Definition 1.5.16. A chain complex A_* is called split if there exists a family of morphisms $(s_n\colon A_n \to A_{n+1})_{n\in\mathbb{Z}}$ satisfying $d_n = d_n s_n d_n$. A split chain complex that is exact is called split exact.

Example 1.5.17. Let k be a field. We also write Vect_k for LMod_R. Every chain complex in Vect_k splits.

1.5.2 Chain homotopy

Definition 1.5.18.

(1) Let $f_*\colon C_* \to D_*$ be a chain map. A null homotopy of f_* consists of a sequence of morphisms $(H_n\colon C_n \to D_{n+1})_{n\in\mathbb{Z}}$ satisfying that

$$f_n = dH_n + H_{n-1}d\colon C_n \to D_n$$

for all $n \in \mathbb{Z}$. If f_* admits a null homotopy, we call f_* null-homotopic.

(2) Let $f_*, g_*\colon C_* \to D_*$ be chain maps. Then a chain homotopy from f_* to g_* is a null homotopy of $f_* - g_*$. If there is a chain homotopy from f_* to g_*, we call f_* and g_* chain homotopic. Note that this is an equivalence relation on chain maps.

Lemma 1.5.19. *Let $f_*, g_*\colon A_* \to B_*$ be two chain homotopic chain maps. Then $H_n(f_*) = H_n(g_*)$ for all $n \in \mathbb{Z}$. In particular, if f_* is null-homotopic, then $H_n(f_*) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Since f_* is chain homotopic to g_* if and only if $f_* - g_*$ is null homotopic, and $H_n(f_* - g_*) = H_n(f_*) - H_n(g_*)$, it suffices to consider the case that f_* is nullhomotopic and show that $H_n(f_ast) = 0$. Let $[x] \in H_n(A_*)$, represented by an n-cycle $x \in Z_n(A_*)$. Then $f_n([x]) = [dH_n(x) - H_{n-1}d(x)] = [dH_n(x)] - [H_{n-1}d(x)] = 0 - 0$, since $dH_n(x)$ is a boundary and $d(x) = 0$ by assumption. $\square$

Definition 1.5.20. A chain map $f_*\colon A_* \to B_*$ is a called a chain homotopy equivalence if there exists a chain map $g_*\colon B_* \to A_*$ such that $g_* \circ f_*$ is chain homotopic to id_{A_*} and $f_* \circ g_*$ is chain homotopic to id_{B_*}.

Any chain homotopy equivalence is a quasi-isomorphism by Lemma 1.5.19. We can thus consider chain homotopy equivalence between chain complexes as a refinement of the notion of two chain complexes being quasi-isomorphic.

1.5.3 Long exact sequences

Theorem 1.5.21. *Let $\mathcal{C}$ be an abelian category and*

$$0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$$

a short exact sequence in $\mathrm{Ch}(\mathcal{C})$*. Then there exists an exact chain complex*

$$\dots \xrightarrow{H_{n+1}(g_*)} H_{n+1}(C_*) \xrightarrow{\partial_{n+1}} H_n(A_*) \xrightarrow{H_n(f_*)} H_n(B_*) \xrightarrow{H_n(g_*)} H_n(C_*) \xrightarrow{\partial_n} H_{n-1}(A_*) \to \dots$$

called the associated long exact sequence.

To prove Theorem 1.5.21, we will apply the Snake Lemma:

Lemma 1.5.22 (Snake lemma)**.** *Consider a commutative diagram in an abelian category* $\mathcal{C}$

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r, "g"] & C' &
\end{tikzcd}$$

whose rows are exact. Then there exists an exact sequence in $\mathcal{C}$

$$\ker(f) \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\partial} \mathrm{coker}(\alpha) \to \mathrm{coker}(\beta) \to \mathrm{coker}(\gamma) \to \mathrm{coker}(g)\,.$$

Proof. Exercise. □

Proof of Theorem 1.5.21. We may assume that the objects of $\mathcal{C}$ have elements, see Theorem 1.4.24.

We first apply the snake lemma to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

yielding the exact sequence

$$0 \to Z_n(A_*) \to Z_n(B_*) \to Z_n(C_*) \to A_{n-1}/dA_n \to B_{n-1}/dB_n \to C_{n-1}/dC_n \to 0$$

where we denote $dA_n = B_{n-1}(A_*)$ for brevity. We rearrange this into the diagram

$$\begin{tikzcd}
 & A_n/dA_{n+1} \arrow[r] \arrow[d, "d"] & B_n/dB_{n+1} \arrow[r] \arrow[d, "d"] & C_n/dC_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(A_*) \arrow[r] & Z_{n-1}(B_*) \arrow[r] & Z_{n-1}(C_*) &
\end{tikzcd}$$

where the vertical morphisms are induced by the differential, using the universal property of the cokernel A_n/dA_{n+1} and that $d|_{dA_{n+1}} = 0$. Let $[x] \in A_n/dA_{n+1} \to$

$Z_{n-1}(A_*)$. Then $d([x]) = [d(x)] = 0$ if any only if $d(x) = 0$, since $d^2 = 0$. Thus, the kernel of d is given by the quotient $Z_n(A_*)/dA_{n+1} = H_n(A_*)$. The image of $d\colon A_n/dA_{n+1} \to A_{n-1}$ is isomophic to the image of $d\colon A_n \to A_{n-1}$, so that the cokernel is given by $H_{n-1}(A_*)$. Applying the snake lemma again, we thus obtain the exact sequence

$$H_n(A_*) \to H_n(B_*) \to H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \to H_{n-1}(B_*) \to H_{n-1}(C_*)\,.$$

Patching these exact sequences together for all $n \in \mathbb{Z}$ yields the desired long exact sequence. $\square$

Remark 1.5.23. The morphism $\partial_n\colon H_n(C_*) \to H_{n-1}(A_*)$ in Theorem 1.5.21 is called the connecting morphism. Given a class $[x] \in H_n(C_*)$, with $x \in Z_n(C_*)$, we can describe $\partial([x]) \in H_{n-1}(A_*)$ as follows: We choose a preimage $y \in B_n$ of x under the epimorphism g_n. We have $d(y) \in \ker(g_{n-1}) = \mathrm{Im}(f_{n-1})$, and we let $z \in Z_{n-1}(A_*)$ be a preimage of $d(y)$ under f_{n-1}. Then $\partial([x]) = [z] \in H_{n-1}(A_*)$.

The connecting homomorphisms satisfies a property called *naturality.* This is a mathematical term, referring to the following notion:

Definition 1.5.24. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta\colon F \to G$ consists of a family of morphisms $(\eta_x\colon F(x) \to G(x))_{x\in\mathcal{C}}$, satisfying that, for any morphism $f\colon x \to y$ in $\mathcal{C}$, the square in $\mathcal{D}$

$$\begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\eta_y"] & G(y) \end{tikzcd}$$

commutes.

Definition 1.5.25. Let $\mathcal{C}, \mathcal{D}$ be two categories. The functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category determined as follows:

- Its objects are given by functors $F\colon \mathcal{C} \to \mathcal{D}$.
- Its morphisms are given by natural transformations $\eta\colon F \to G$.
- The identity morphisms are the identity natural transformations

$$\mathrm{id}_F = (\mathrm{id}_{F(x)})_{x\in\mathcal{C}}\colon F \to F\,.$$

- The composition of two natural transformations $\eta\colon F \to G$ and $\mu\colon G \to H$ is given by the natural transformation $\mu \circ \eta\colon F \to H$ corresponding to the family of morphisms $(\mu_x \circ \eta_x)_{x\in\mathcal{C}}$.

Example 1.5.26. Let $\mathcal{C}$ be an abelian category

(1) Consider the poset category $(\mathbb{Z}, \leq)$ (meaning morphisms are of the form $i \to j$ with $i \leq j$). The category $\mathrm{Ch}(\mathcal{C})$ is isomorphic to the full subcategory of $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$ of functors $F\colon \mathbb{Z}^{\mathrm{op}} \to \mathcal{C}$ satisfying that $F(j \to i) = 0$ for any $i, j \in \mathbb{Z}$ with $j > i + 1$. This isomorphism of categories identifies chain maps with natural transformation between functors $\mathbb{Z}^{\mathrm{op}} \to \mathcal{C}$.

(2) We denote by $\mathrm{LES}(\mathcal{C}) \subset \mathrm{Ch}(\mathcal{C})$ the full subcategory of exact chain complexes.

(3) Consider the full subcategory $[2] = \{0 \to 1 \to 2\} \subset \mathbb{Z}$. We define $\mathrm{SES}(\mathcal{C}) \subset \mathrm{Fun}([2]^{\mathrm{op}}, \mathcal{C})$ as the full subcategory of short exact sequences in $\mathcal{C}$.

Proposition 1.5.27. *Let $\mathcal{C}$ be an abelian category. There exists a functor*

$$\mathrm{SES}(\mathrm{Ch}(\mathcal{C})) \to \mathrm{LES}(\mathcal{C}),$$

associating to each short exact sequence of chain complexes its associated long exact sequence from Theorem 1.5.21.

Proof. We need to show that for each morphism in $\mathrm{SES}(\mathrm{Ch}(\mathcal{C}))$, corresponding to a commutative diagram in $\mathrm{Ch}(\mathcal{C})$ with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "f_*"] \arrow[d, "\alpha_*"] & B_* \arrow[r, "g_*"] \arrow[d, "\beta_*"] & C_* \arrow[r] \arrow[d, "\gamma_*"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "f'_*"] & B'_* \arrow[r, "g'_*"] & C'_* \arrow[r] & 0 \end{tikzcd}$$

there is a corresponding chain map between the associated long exact sequences:

$$\begin{tikzcd} \dots \arrow[r] & H_n(A_*) \arrow[r] \arrow[d, "H_n(\alpha_*)"] & H_n(B_*) \arrow[r] \arrow[d, "H_n(\beta_*)"] & H_n(C_*) \arrow[r, "\partial"] \arrow[d, "H_n(\gamma_*)"] & H_{n-1}(A_*) \arrow[r] \arrow[d, "H_{n-1}(\alpha_*)"] & \dots \\ \dots \arrow[r] & H_n(A'_*) \arrow[r] & H_n(B'_*) \arrow[r] & H_n(C'_*) \arrow[r, "\partial"] & H_{n-1}(A'_*) \arrow[r] & \dots \end{tikzcd}$$

The commutativity of the first two squares follows from H_n(-) being a functor. We verify that the right square commutes on elements. Let $x \in Z_n(C_*)$ and $y \in B_n$ with $g_n(y) = x$. Then $\beta_n(y) \in Z_n(B'_*)$ is the preimage of $\gamma_n(x) \in Z_n(C'_*)$ since $g'_n \circ \beta_n(y) = \gamma_n \circ g_n(y) = \gamma_n(x)$. Let $z \in Z_{n-1}(A_*)$ with $f_{n-1}(z) = d(y) \in Z_{n-1}(B_*)$. Then $\alpha_{n-1}(z) \in Z_{n-1}(A'_*)$ satisfies

$$f'_{n-1} \circ \alpha_{n-1}(z) = \beta_{n-1} \circ f_{n-1}(z) = \beta_{n-1} \circ d(y) = d \circ \beta_n(y),$$

meaning that $\alpha_{n-1}(z)$ is the preimage of $d \circ \beta_n(y)$. This shows that $[\alpha_{n-1} \circ \partial(x)] = [\alpha_{n-1}(z)] = [\partial \circ \gamma_n(x)]$, expressing the desired commutativity of the right square. $\square$

Definition 1.5.28. Let $\mathcal{C}$ be an abelian category and $f_*\colon C_* \to D_*$ a morphism in $\mathrm{Ch}(\mathcal{C})$. The cone of f_*, denoted $\mathrm{cone}(f_*) \in \mathrm{Ch}(\mathcal{C})$, is defined as the chain complex with

- n-chains given by $\mathrm{cone}(f_*)_n = D_n \oplus C_{n-1}$ for all $n \in \mathbb{Z}$,
- and differential $d_n \colon D_n \oplus C_{n-1} \to D_{n-1} \oplus C_{n-2}$ corresponding to the matrix[2]

$$d = \begin{pmatrix} d_n^D & -f_{n-1} \\ 0 & -d_{n-1}^C \end{pmatrix}$$

Let us check that $d^2 = 0$, so that $\mathrm{cone}(f_*)$ is indeed a chain complex:

$$d^2 = \begin{pmatrix} d_n^D & -f_{n-1} \\ 0 & -d_{n-1}^C \end{pmatrix} \circ \begin{pmatrix} d_{n+1}^D & -f_n \\ 0 & -d_n^C \end{pmatrix} = \begin{pmatrix} d^2 & -f_{n-1} \circ d_n^D + d_{n-1}^C \circ f_n \\ 0 & d^2 \end{pmatrix} = 0$$

The last equality uses that f_* is a chain map.

Definition 1.5.29. Let C_* be a chain complex and $n \in \mathbb{Z}$. Then the shifted chain complex $C_*[n]$ is defined as the chain complex with

$$(C_*[n])_a = C_{a-n}$$

and

$$d_a^{\mathcal{C}_*[n]} = (-1)^n d_{a-n}^{C_*}$$

for all $a \in \mathbb{Z}$.

Note that $C_*[1] \simeq \mathrm{cone}(C_* \to 0)$.

Example 1.5.30. Let $\mathcal{C}$ be an abelian category and $f_* \colon C_* \to D_*$ a chain map in $\mathcal{C}$. There exists a short exact sequence in $\mathrm{Ch}(\mathcal{C})$

$$0 \to D_* \to \mathrm{cone}(f_*) \to C_*[1] \to 0\,.$$

Theorem 1.5.21 implies that $\mathrm{cone}(f_*)$ is exact if and only if f_* is a quasi-isomorphism.

1.6 Tensor products

1.6.1 The tensor product of abelian groups

Definition 1.6.1. Let A, B, C be abelian groups. We call a morphism of sets $f(\text{-},\text{-})\colon : A \times B \to C$ bilinear (or $\mathbb{Z}$-bilinear) if

- $f(\text{-}, b)\colon A \to C$ is a morphism of abelian groups for all $b \in B$, and
- $f(a, \text{-})\colon B \to C$ is a morphism of abelian groups for all $a \in A$.

Note that a bilinear morphism is typically not a morphism of abelian groups.

[2]For this, we argue as in Remark 1.4.9 and consider a 2-vector of 2-vectors as a 2×2-matrix.

Definition 1.6.2 (The universal property of the tensor product)**.** Let A, B be two abelian groups. An abelian group C, equipped with a bilinear morphism

$$\pi\colon A \times B \to C\,,$$

is called the tensor product of A, B if for every abelian group D and bilinear morphism $f\colon A \times B \to D$ there exists a unique morphism of abelian groups $f'\colon C \to D$ (also referred to as a $\mathbb{Z}$-linear morphism) satisfying that

$$f' \circ \pi = f\,.$$

In this case, we write $C = A \otimes B$.

We can illustrate the universal property of the tensor product via the following diagram:

$$\begin{tikzcd} A \times B \arrow[r, "\mathbb{Z}\text{-bilin.}"] \arrow[d, "\pi"'] & D \\ A \otimes B \arrow[ur, "\exists!\mathbb{Z}\text{-linear}"'] & \end{tikzcd}$$

Note that the universal property of the tensor product is formulated not within the category Ab: it does not only involve objects and morphisms in Ab. Nevertheless, the tensor product is unique up to unique isomorphism in Ab:

Lemma 1.6.3 (Uniqueness of the tensor product)**.** *Let A, B be abelian groups and $\pi_1\colon A \times B \to C_1$ and $\pi_2\colon A \times B \to C_2$ two bilinear maps exhibiting both C_1 and C_2 as the tensor product of A and B. Then there exists a unique isomorphism of abelian groups $\phi\colon C_1 \to C_2$ satisfying $\pi_2 = \phi \circ \pi_1$.*

Proof. A similar argument as in the proof of Lemma 1.3.16 applies. □

We next show that the tensor product of any two abelian groups exists. We will construct it as a quotient of a free abelian group.

Notation 1.6.4. Given a set M, we denote by $F(M) = \mathbb{Z}^{\amalg M} \in \mathrm{Ab}$ the free abelian group generated by F. Given an abelian group A, we denote by $F(A)$ the free abelian group on the set underlying A.

Given a pair of abelian groups A, B, we can write elements of $F(A \times B)$ as finite sums of the free generating set $\{(a, b) | a \in A, b \in B\}$. We let $R \subset F(A \times B)$ be the subgroup generated by elements of following form:

- $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$, with $a_1, a_2 \in A$, $b \in B$, and
- $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$, with $a \in A$, $b_1, b_2 \in B$.

Lemma 1.6.5. *Let A, B be two abelian groups. There exists a bilinear map*

$$\pi\colon A \times B \to F(A \times B)/R, \quad (a, b) \mapsto [(a, b)]$$

exhibiting $F(A \times B)/R$ as the tensor product of A and B.

Proof. We first verify that π is indeed bilinear. Let $a_1, a_2 \in A$ and $b \in B$. Then

$$\pi(a_1+a_2, b) = [(a_1+a_2, b)] = [(a_1, b)+(a_2, b)] = [(a_1, b)]+[(a_2, b)] = \pi(a_1, b)+\pi(a_2, b)\,,$$

where the second equality follows from the relation $[(a, b_1 + b_2) - (a, b_1) - (a, b_2)] = 0$ imposed by quotienting by R. A similar argument shows that $\pi(a, b_1 + b_2) = \pi(a, b_1) + \pi(a, b_2)$ for all $a \in A$ and $b_1, b_2 \in B$, so that we conclude that π is bilinear.

Let $f\colon A \times B \to D$ be a bilinear morphism. We next show the desired unique factorization of f through π.

Let $f', f''\colon F(A \times B)/R \to D$ be two morphisms of abelian groups satisfying that $f' \circ \pi = f$ and $f'' \circ pi = f$. Then for $(a, b) \in A \times B$, we have $f''([(a, b)]) = f'([(a, b)]) = f(a, b)$. The elements $\{(a, b)|a \in A, b \in B\} \subset F(A \times B)$ generate $F(A \times B)$ and so the elements $\{[(a, b)]|a \in A, b \in B\} \subset F(A \times B)/R$ also generate $F(A \times B)/R$. Thus, for any element $x \in F(A \times B)/R$, we have $x = \sum_{i \in I}[(a_i, b_i)]$ for some finite set I and $a_i \in A, b_i \in B$ for all $i \in I$, so that

$$f'(x) = \sum_{i \in I} f(a_i, b_i) = f''(x)\,.$$

This shows that $f' = f''$.

It thus remains to show that a factorization f' satisfying $f' \circ \pi = f$ exists. Using the universal property of the free abelian group $F(A \times B)$ (i.e. free $\mathbb{Z}$-module), see Proposition 1.1.10, the morphism of sets $f\colon A \times B \to D$ determines a morphism of abelian groups $\tilde{f}\colon F(A \times B) \to D$. For $a_1, a_2 \in A$ and $b \in B$, we have

$$\tilde{f}((a_1 + a_2, b)) = f((a_1 + a_2, b)) = f((a_1, b)) + f((a_2, b)) = \tilde{f}((a_1, b) + (a_2, b))\,,$$

where the second equality follows from f being bilinear. It follows that

$$\tilde{f}((a_1 + a_2, b) - (a_1, b) + (a_2, b)) = \tilde{f}((a_1 + a_2, b)) - \tilde{f}((a_1, b) + (a_2, b)) = 0$$

where the first equality follows from $\tilde{f}$ being a morphism of abelian groups. Similarly, one shows that $\tilde{f}((a, b_1 + b_2) - (a, b_1) + (a, b_2)) = 0$ for all $a \in A$ and $b_1, b_2 \in B$.

This show that the composite of $\tilde{f}$ with the inclusion $R \subset F(A \times B)$ vanishes. By the universal property of the cokernel $F(A \times B)/R$, there exists a unique morphism of abelian groups $f'\colon F(A \times B)/R \to D$ satisfying that the following diagram commutes:

$$\begin{tikzcd}
A \times B \arrow[d, mapsto, "{(a,b)\mapsto(a,b)}"] \arrow[drr, "f"] \arrow[dd, bend right, "\pi"'] & & \\
F(A \times B) \arrow[rr, "\tilde{f}"] \arrow[d, two heads] & & D \\
F(A \times B)/R \arrow[urr, dashed, "\exists! f'"'] & &
\end{tikzcd}$$

We thus have $f' \circ \pi = f$, as desired. $\square$

Lemma 1.6.6. *The tensor products forms a functor*

$$(\text{-}) \otimes (\text{-})\colon\ \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}\ .$$

Proof. Given $f\colon A \to A'$ and $g\colon B \to B'$, we obtain a bilinear morphism $A \times B \xrightarrow{(f,g)} A' \times B' \xrightarrow{\pi} A' \otimes B'$, inducing a morphism $f \otimes g\colon A \otimes B \to A' \otimes B'$. Using the universal property of the tensor product, one readily checks that this extends to the desired functor $(\text{-}) \otimes (\text{-})$. $\square$

Example 1.6.7. The bilinear morphism of abelian groups

$$\pi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (a,b) \mapsto a \cdot b$$

exhibits $\mathbb{Z}$ as the tensor product $\mathbb{Z} \otimes \mathbb{Z}$:

Let D be an abelian group and $f(\text{-},\text{-})\colon \mathbb{Z} \times \mathbb{Z} \to D$ a bilinear morphism. Then $f(\text{-},\text{-})$ is a $\mathbb{Z}$-module homomorphism in each entry and thus for any $i, j \in \mathbb{Z}$, we have

$$f(i,j) = i \cdot j \cdot f(1,1)\,.$$

Any morphism of abelian groups $f'\colon \mathbb{Z} \to D$ is determined by $f'(1) \in D$ via the relation $f'(i) = i \cdot f'(1)$. The condition

$$f(i,j) = i \cdot j \cdot f(1,1) = f' \circ \pi(i,j) = f'(i \times j) = i \cdot j \cdot f'(1)$$

shows that $f = f' \circ \pi$ holds if and only if $f'(1) = f(1,1)$. This shows the desired unique factorizaton.

Remark 1.6.8. We will prove further below that $\mathbb{Z}^{\amalg I} \otimes \mathbb{Z}^{\amalg J} \simeq \mathbb{Z}^{\amalg(I \times J)}$ for any two sets I, J.

Consider two abelian groups $A \otimes B$ with tensor product $\pi\colon A \times B \to A \otimes B$. Each pair $(a,b) \in A \otimes B$ determines an element $a \otimes b := \pi(a,b) \in A \otimes B$. Elements of this form are sometimes referred to as *pure tensors* in $A \otimes B$. Not every element of $A \otimes B$ must be of this form. The construction of $A \otimes B$ in Lemma 1.6.5 however shows that every element of $A \otimes B$ can be written as a finite sum of pure tensors.

1.6.2 The tensor product of bimodules

Remark 1.6.9. Given a ring R and a left R-module M, the left action $R \times M \to M$ is bilinear. It thus determines (and is determined by) a morphism of abelian groups $\mu\colon R \otimes M \to M$. If M is instead a right module, we similarly denote the induced morphism $\mu\colon M \otimes R \to M$ by μ.

Definition 1.6.10. A coequalizer in a category is the colimit of a diagram indexed by the category Z with two objects $0, 1$ and $\mathrm{Hom}_Z(i,i) = \{\mathrm{id}_Z\}$, $i = 0, 1$, $\mathrm{Hom}_Z(0,1) = * \amalg *$ and $\mathrm{Hom}_Z(1,0) = \emptyset$.

$$Z = \quad 0 \rightrightarrows 1$$

We note that Ab admits colimits by Lemma 1.3.21 and thus also coequalizers.

Definition 1.6.11. Let R be a ring, M a right R-module and N a left R-module. Then R-linear tensor product $M \otimes_R N \in \mathrm{Ab}$ is defined as the tip of the following coequalizer cocone in Ab

$$M \otimes R \otimes N \underset{\mu \otimes \mathrm{id}_N}{\overset{\mathrm{id}_M \otimes \mu}{\rightrightarrows}} M \otimes N \xrightarrow{\ p\ } M \otimes_R N$$

Given $x \in M$, $y \in N$, we also write $x \otimes_R y := p(x \otimes y) \in M \otimes_R N$.

We note that the morphism p in Definition 1.6.11 is an epimorphism.

Remark 1.6.12. In the setting of Definition 1.6.11, let $r \in R, x \in M, y \in N$. Then

$$(x \cdot r) \otimes_R y = x \otimes_R (r \cdot y)\,.$$

This follows from evaluating the equality of morphisms $p \circ (\mu \otimes \mathrm{id}_N) = p \circ (\mathrm{id}_M \otimes \mu)$ at $x \otimes r \otimes y \in M \otimes R \otimes N$.

Example 1.6.13. Every abelian group is a left and right $\mathbb{Z}$-module. For two abelian groups A, B, we have $A \otimes_{\mathbb{Z}} B \simeq A \otimes B$.

In the case that R is commutative, we can equip the R-relative tensor product with an R-module structure. This will follow from a more general construction involving bimodules.

Definition 1.6.14. Let R, S be two rings. An R-S-bimodule consists of an abelian group M, together with a left R-action $R \times M \to M$ and a right S-action $M \times S \to M$, exhibiting it as a left R-module and a right S-module, satisfying that

$$(r \cdot m) \cdot s = r \cdot (m \cdot s)$$

for all $r \in R$, $m \in M$, and $s \in S$.

Definition 1.6.15. Given two ring R, S, we denote by ${}_R\mathrm{BiMod}_S$ the category with objects given by R-S-bimodules and morphisms given by morphisms of abelian groups which commute with both the left R-action and the right S-action.

Remark 1.6.16. The category ${}_R\mathrm{BiMod}_S$ is abelian and the forgetful functors ${}_R\mathrm{BiMod}_S \to \mathrm{LMod}_R$ and ${}_R\mathrm{BiMod}_S \to \mathrm{RMod}_S$ are exact.

Example 1.6.17. Let R be a commutative ring. Then each left R-module M admits a canonical structure of an R-R-bimodule: Since R is commutative, $R = R^{\mathrm{op}}$, and so M admits a right R-action $M \times R \to M$, given by $m \cdot r := r \cdot m$ for $r \in R$ and $m \in M$. These two actions define the R-R-bimodule structure on M: for $r_1, r_2 \in R$ and $m \in M$, we have

$$\begin{aligned} r_1 \cdot (m \cdot r_2) &= r_1 \cdot (r_2 \cdot m) \\ &= (r_1 r_2) \cdot m \\ &= (r_2 r_1) \cdot m \\ &= r_2 \cdot (r_1 \cdot m) \\ &= (r_1 \cdot m) \cdot r_2\,. \end{aligned}$$

Equipping a left R-module with the above R-R-bimodule structure defines an exact fully faithful functor $\mathrm{LMod}_R \hookrightarrow {}_R\mathrm{BiMod}_R$. The category LMod_R is isomorphic to the full subcategory of ${}_R\mathrm{BiMod}_R$ consisting of R-R-bimodules M, where $r \cdot m = m \cdot r$ for all $m \in M$ and $r \in R$.

Lemma 1.6.18. *Let R, S be rings. Let $A \in \mathrm{LMod}_R$ and $B \in \mathrm{RMod}_S$. Then there exists a canonical R-S-bimodule structure on $A \otimes B$, satisfying that $r \cdot (x \otimes y) = (r \cdot x) \otimes y$ and $(x \otimes y) \cdot s = x \otimes (y \cdot s)$ for $r \in R$, $s \in S$, $x \in A$, and $y \in B$.*
Further, any two morphism $f \colon A \to A'$ in LMod_R and $g \colon B \to B'$ in RMod_S extend to a morphism of R-S-bimodules $A \otimes B \xrightarrow{f \otimes g} A' \otimes B'$.

Proof. We only describe the right S-action on $A \otimes B$, the construction of the left R-action is similar. Let $s \in S$. The morphism $(\text{-}) \cdot s \colon B \to B, b \mapsto b \cdot s$ is linear. Hence, the composed morphism

$$A \times B \xrightarrow{\mathrm{id}_A \times (\text{-}) \cdot s} A \times B \xrightarrow{\pi} A \otimes B$$

is bilinear. Using the universal property of the tensor product, this determines a morphism of abelian groups, denoted $\mathrm{id}_A \otimes ((\text{-}) \cdot s) \colon A \otimes B \to A \otimes B$. We define the morphism of sets $m(\text{-},\text{-}) \colon (A \otimes B) \times S \to A \otimes B$, by the assignment $(x, s) \mapsto m(x, s) = \mathrm{id}_A \otimes ((\text{-}) \cdot s)(x)$. By construction, $m(\text{-},\text{-})$ is linear in the first entry. Given $s, s' \in S$, we have

$$(\text{-}) \cdot s + (\text{-}) \cdot s' = (\text{-}) \cdot (s + s') \colon B \to B$$

and thus also $\mathrm{id}_A \otimes ((\text{-}) \cdot s) + \mathrm{id}_A \otimes ((\text{-}) \cdot s') = \mathrm{id}_A \otimes ((\text{-}) \cdot (s + s'))$. Evaluating this equality at any $x \in A \otimes B$ shows that $m(\text{-},\text{-})$ is linear in the second entry.

The left R-action and the right S-action on $A \otimes B$ are compatible in the sense of defining a R-S-bimodule structure: on pure tensors, we have

$$(r \cdot (x \otimes y)) \cdot s = ((r \cdot x) \otimes y) \cdot s = (r \cdot x) \otimes (y \cdot s) = r \cdot ((x \otimes y) \cdot s)\,.$$

Since pure tensors generated $A \otimes B$, the compatibility of actions also holds on arbitrary elements of $A \otimes B$.

The morphism $f \otimes g \colon A \otimes B \to A' \otimes B'$ is constructed using the universal property of the tensor product. That it commutes with the left R-action and the right S-action can be checked on pure tensors, where $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$, and the compatibility with the actions is clear. $\square$

Remark 1.6.19. Let R, S, T be rings.

(1) Let $M \in {}_R\mathrm{BiMod}_S$. The compatibility of the left R-action and the right S-action on M implies that the morphism $\mu \colon R \otimes M \to M$ lifts to a morphism in RMod_S.

(2) Let $M \in {}_R\mathrm{BiMod}_S$ and $N \in {}_S\mathrm{BiMod}_T$. Applying Lemma 1.6.18 in the case $A = M$, $B = S \otimes N$, $B' = N$ and $f = \mu \colon S \otimes N \to N$, we find that the morphism of abelian groups $\mathrm{id}_M \otimes \mu \colon M \otimes S \otimes N \to M \otimes N$ lifts to a morphism in ${}_R\mathrm{BiMod}_T$. A similar statement holds for $\mu \otimes \mathrm{id}_N$.

Definition 1.6.20. Let R, S, T be rings, $M \in {}_R\mathrm{BiMod}_S$ and $N \in {}_S\mathrm{BiMod}_T$. Then $M \otimes_S N \in {}_R\mathrm{BiMod}_T$ is defined as the tip of the following coequalizer cocone in ${}_R\mathrm{BiMod}_T$:

$$M \otimes S \otimes N \underset{\mu \otimes \mathrm{id}_N}{\overset{\mathrm{id}_M \otimes \mu}{\rightrightarrows}} M \otimes N \xrightarrow{p} M \otimes_S N \tag{13}$$

Remark 1.6.21. Let R be a commutative ring, $M \in \mathrm{RMod}_R \subset {}_R\mathrm{BiMod}_R$ and $N \in \mathrm{RMod}_R \simeq \mathrm{LMod}_R \subset {}_R\mathrm{BiMod}_R$. Then the R-relative tensor product $M \otimes_R N \in {}_R\mathrm{BiMod}_R$ lies in the full subcategory $\mathrm{LMod}_R \subset {}_R\mathrm{BiMod}_R$. The relative tensor product $M \otimes_R N$ is in this case considered as an R-module, i.e. $M \otimes_R N \in \mathrm{LMod}_R$.

Unraveling the construction of the relative tensor product over a commutative ring, one finds the following universal property. This may be reminiscent to linear algebra, where the tensor product of vector space (i.e. modules over a field) was introduced this way.

Proposition 1.6.22 (Universal property of the tensor product over a commutative ring)**.** *Let R be a commutative ring and $M, N \in \mathrm{LMod}_R$. Then there exists a R-bilinear morphism $\pi\colon M \times N \to M \otimes_R N$ such that for any other R-bilinear morphism $f\colon M \times N \to D$, with $D \in \mathrm{LMod}_R$, there exists a unique morphism $f'\colon M \otimes_R N \to D$ in LMod_R satisfying that $f' \circ \pi = f$.*

Proof. Omitted. ☐

Lemma 1.6.23. *Let R, S, T be rings. Then the S-relative tensor product forms a functor*

$$(\text{-}) \otimes_S (\text{-})\colon\ {}_R\mathrm{BiMod}_S \times {}_S\mathrm{BiMod}_T \to {}_R\mathrm{BiMod}_T\ .$$

Specializing to $T = \mathbb{Z}$, an R-S-bimodule M thus defines a functor

$$M \otimes_S (\text{-})\colon\ \mathrm{LMod}_S \to \mathrm{LMod}_R\ .$$

Proof sketch. Using the functoriality of the tensor product of abelian groups (1.6.6), we obtain a morphism between the coequalizer diagrams, see (13), which induces a morphism between the coequalizers. This morphisms gives rise to the functoriality. ☐

1.6.3 Adjunctions

Definition 1.6.24. A natural transformation $\eta\colon F \to G$ is called a natural isomorphism if there exists a natural transformation $\mu\colon G \to F$ such that $\mu \circ \eta \simeq \mathrm{id}_F$ and $\eta \circ \mu \simeq \mathrm{id}_G$.

Remark 1.6.25. A natural transformation $\eta\colon F \to G$, with $F, G\colon \mathcal{C} \to \mathcal{D}$, is a natural isomorphism if and only if $\eta_x\colon F(x) \to G(x)$ is an isomorphism in $\mathcal{D}$ for all $x \in \mathcal{C}$.

Definition 1.6.26. Let $\mathcal{C}, \mathcal{D}$ be a categories. A pair of functors $F\colon \mathcal{C} \longleftrightarrow \mathcal{D} : G$ is called an adjunction if there exists a natural isomorphism of functors

$$\Phi\colon \operatorname{Hom}_{\mathcal{D}}(F^{\mathrm{op}}(\text{-}),\text{-}) \simeq \operatorname{Hom}_{\mathcal{C}}(\text{-}, G(\text{-}))\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set} \;.$$

In this case, we also call F the left adjoint of G, G the right adjoint of F, and write $F \dashv G$. Given $x \in \mathcal{C}$ and $y \in \mathcal{D}$, we denote by

$$\Phi_{x,y}\colon \operatorname{Hom}_{\mathcal{D}}(F(x), y) \simeq \operatorname{Hom}_{\mathcal{C}}(x, G(y))$$

the arising bijection of Hom sets.

Example 1.6.27. Let R be a ring. Then there is an adjunction

$$R^{\oplus(\text{-})}\colon \mathrm{Set} \longleftrightarrow \mathrm{LMod}_R \;: U$$

where U is the forgetful functor, mapping an R-module to its underlying set. This amounts to the statement that the bijection

$$\operatorname{Hom}_{\mathrm{LMod}_R}(R^{\oplus I}, M) \simeq \operatorname{Hom}_{\mathrm{Set}}(I, U(M))$$

from Proposition 1.1.10 is natural in I and M.

Remark 1.6.28. The naturality of Φ in Definition 1.6.26 amounts to the commutativity of the following two squares for all $f\colon x \to x'$ and $g\colon y \to y'$:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{D}}(F(x'), y) \arrow[r, "\Phi_{x',y}"] \arrow[d, "(\text{-})\circ F(f)"'] & \operatorname{Hom}_{\mathcal{C}}(x', G(y)) \arrow[d, "(\text{-})\circ f"] \\
\operatorname{Hom}_{\mathcal{D}}(F(x), y) \arrow[r, "\Phi_{x,y}"] & \operatorname{Hom}_{\mathcal{C}}(x, G(y))
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{D}}(F(x), y) \arrow[r, "\Phi_{x,y}"] \arrow[d, "g\circ(\text{-})"'] & \operatorname{Hom}_{\mathcal{C}}(x, G(y)) \arrow[d, "G(g)\circ(\text{-})"] \\
\operatorname{Hom}_{\mathcal{D}}(F(x), y') \arrow[r, "\Phi_{x,y'}"] & \operatorname{Hom}_{\mathcal{C}}(x, G(y'))
\end{tikzcd}$$

Evaluating the first square at a morphism $h\colon F(x') \to y$, we obtain

$$\Phi(h) \circ f = \Phi(h \circ F(f)) \tag{14}$$

where we have omitted the indices of Φ to simplify the expression. Similarly, evaluating the second square at $h\colon F(x) \to y$, we obtain

$$G(g) \circ \Phi(h) = \Phi(g \circ h)\,. \tag{15}$$

We next describe an equivalent characterization of adjunctions, in terms of unit and counit natural transformations. For this, we will need to introduc the whiskering of natural transformations.

Remark 1.6.29. Let $F, F' \colon \mathcal{C} \to \mathcal{D}$, $E \colon \mathcal{B} \to \mathcal{C}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be a functor. Given a natural transformation $\eta \colon F \to F'$, we can produce a new natural transformation $\eta * E \colon FE \to F'E$, given by the family of morphisms $\{\eta_{E(x)} \colon FE(x) \to F'E(x)\}_{x \in \mathcal{B}}$. Similarly, we can produce a natural transformation $G * \eta \colon G \circ F \to G \circ F'$, given by the family of morphisms $\{G(\eta_y) \colon GF(y) \to GF'(y)\}_{y \in \mathcal{C}}$. These natural transformations $\eta * E, G * \eta$ are also referred to as the *whiskering* of η and E or G.

Definition 1.6.30. Let $\mathcal{C}, \mathcal{D}$ be a categories and consider a pair of functor $F \colon \mathcal{C} \longleftrightarrow \mathcal{D} : G$. A pair of natural transformation $\epsilon \colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\eta \colon FG \to \mathrm{id}_{\mathcal{D}}$ are called the unit and counit of F, G if together they satisfying

$$(\eta * F) \circ (F * \epsilon) = \mathrm{id}_F \tag{16}$$

and

$$(G * \eta) \circ (\epsilon * G) = \mathrm{id}_G \ . \tag{17}$$

The two conditions (16) and (17) are referred to as the *triangle identities* and can be depicted as follows:

$$\begin{tikzcd} & FGF \arrow[dr, "\eta * F"] & \\ F \arrow[ur, "F * \epsilon"] \arrow[rr, "\mathrm{id}_F"] & & F \end{tikzcd} \qquad \begin{tikzcd} & GFG \arrow[dr, "G * \eta"] & \\ G \arrow[ur, "\epsilon * G"] \arrow[rr, "\mathrm{id}_G"] & & G \end{tikzcd}$$

Proposition 1.6.31. *A pair of functor $F \colon \mathcal{C} \longleftrightarrow \mathcal{D} : G$ forms an adjunction if and only if there exist unit and counit transformations $\epsilon \colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\eta \colon FG \to \mathrm{id}_{\mathcal{D}}$.*

Proof. We first suppose that F, G form an adjunction with natural equivalence ϕ. We begin by constructing the corresponding unit and counit. For $x \in \mathcal{C}$, define

$$\epsilon_x = \Phi_{x, F(x)}(\mathrm{id}_{F(x)}) \colon x \to GF(x) \, .$$

Then $\epsilon = (\epsilon_x)_{x \in \mathcal{C}} \colon \mathrm{id}_{\mathcal{C}} \to GF$ forms a natural transformation: let $\alpha \colon x \to x'$ be a morphism in $\mathcal{C}$. Then the commutativity of the diagram

$$\begin{tikzcd} x \arrow[r, "\epsilon_x"] \arrow[d, "\alpha"'] & GF(x) \arrow[d, "GF(\alpha)"] \\ y \arrow[r, "\epsilon_y"] & GF(y) \end{tikzcd}$$

follows from

$$GF(\alpha) \circ \Phi(\mathrm{id}_{F(x)}) \overset{(15)}{=} \Phi(F(\alpha)) \overset{(14)}{=} \Phi(\mathrm{id}_{F(x)}) \circ \alpha \, .$$

For $y \in \mathcal{D}$, we define

$$\eta_y = \Phi^{-1}_{G(y), y}(\mathrm{id}_{G(x)}) \colon FG(y) \to y$$

and conclude from a similar argument as above that $\eta = (\eta_y)_{y \in \mathcal{D}} \colon FG \to \mathrm{id}_{\mathcal{D}}$ also forms a natural transformation.

Given a morphism $f\colon F(x) \to y$, Equation (15) states that $\Phi(f) = G(f) \circ \epsilon_x$. Similarly, applying Φ^{-1} to Equation (14), we find that for $g\colon x \to G(y)$, we have $\Phi^{-1}(g) = \eta_y \circ F(g)$. Given a unit and counit for F, G, we can define Φ and Φ^{-1} via these equations. The naturality of Φ then readily follows from the naturality of ϵ.

Finally, it remains to note that Φ and Φ^{-1} are inverse bijections if and only if the triangle identities hold. Indeed, for $f\colon F(x) \to y$, we have

$$\Phi^{-1}(\Phi(f)) = \eta_{G(y)} \circ F(G(f) \circ \epsilon_x) = f \circ \eta_{G(y)} \circ F(\epsilon_x)\,,$$

where the second equality follows from the naturality of $\eta_{G(y)}$, and for $g\colon x \to G(y)$, we similarly have

$$\Phi(\Phi^{-1}(g)) = G(\eta_y \circ F(g)) \circ \epsilon_{F(x)} = G(\eta_y) \circ \epsilon_{F(x)} \circ g\,.$$

□

Definition 1.6.32. Let $F\colon \mathcal{C} \leftrightarrow \mathcal{D} : G$ be an adjunction. The adjunction is called an adjoint equivalence if the unit and counit are natural isomorphisms.

Remark 1.6.33. Consider a pair of functor $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ satisfying that there exists natural equivalences $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$. In this case, F, G are called *equivalences of categories*. If F, G are equivalences of categories, they can be equipped with unit and counit transformations exhibiting them as an adjoint equivalence.

Every isomorphism of categories is an equivalence of categories, but the notion of equivalence of categories is weaker. Isomorphic (small) categories have same sets of objects. Equivalent categories in general do not: the category $\mathrm{Vect}_k^{\mathrm{fd}}$ of finite dimensional vector spaces over a field k is equivalent to its full subcategory given by vector spaces of the form $k^{\oplus n}$, $n \geq 0$.

Remark 1.6.34. If a functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint, the right adjoint is unique up to natural isomorphism. Given two right adjoints G, G', the natural isomorphism between them is given by $G \xrightarrow{\mathrm{u}' * G} G'FG \xrightarrow{G' * \mathrm{cu}} G'$.

Exercise 1.6.35. Consider an adjunction $F\colon \mathcal{C} \longleftrightarrow \mathcal{D} : G$. Then F preserves all colimits which exist in $\mathcal{C}$ and G preserves all limits which exist in $\mathcal{D}$.

1.6.4 The tensor-hom adjunction

Proposition 1.6.36. *(1) Let A be an abelian group. Then there exists an adjunction*

$$(\text{-}) \otimes A\colon\ \mathrm{Ab} \longleftrightarrow \mathrm{Ab}\ : \mathrm{Hom}_{\mathrm{Ab}}(A, \text{-})\,.$$

(2) Let R be a commutative ring and M an R-module. Then there exists an adjunction

$$M \otimes_R (\text{-})\colon\ \mathrm{LMod}_R \longleftrightarrow \mathrm{LMod}_R\ : \mathrm{Hom}_R(M, \text{-})\,.$$

(3) Let R, S be rings and M be an S-R-bimodule. Then there exists an adjunction

$$M \otimes_R (\text{-})\colon \ \mathrm{LMod}_R \longleftrightarrow \mathrm{LMod}_S \ :\mathrm{Hom}_S(M, \text{-})\,.$$

We note that $r \in R$ acts on $f \in \mathrm{Hom}_S(M, N)$ from the left via $(r \cdot f)(m) = f(m \cdot r)$ for all $m \in M$.

Proof. We note that (2) is a special case of (3), obtained from considering the left R-module M as a bimodule as in Example 1.6.17. In turn, part (1) is a special case of (2), corresponding to $R = \mathbb{Z}$, and switching from left modules to right modules.

We will however first prove part (1), and then sketch how part (3) follows from this. Let A, B, C be abelian groups. We show that $(\text{-}) \otimes B \dashv \mathrm{Hom}_{\mathrm{Ab}}(B, \text{-})$. We denote by $\mathrm{BiLin}(A, B; C) \subset \mathrm{Hom}_{\mathrm{Set}}(A \times B, C)$ the subset of bilinear morphisms. The universal property of the tensor product yields a bijection

$$\Phi^1_{A,C}\colon \ \mathrm{Hom}_{\mathrm{Ab}}(A \otimes B; C) \simeq \mathrm{BiLin}(A, B; C)\,.$$

We next describe a bijection

$$\Phi^2_{A,C}\colon \ \mathrm{BiLin}(A, B; C) \simeq \mathrm{Hom}_{\mathrm{Ab}}(A, \mathrm{Hom}_{\mathrm{Ab}}(B, C)).$$

Given a bilinear morphism $f(\text{-},\text{-})\colon A \times B \to C$, we define the morphism of abelian groups

$$\Phi^2_{A,C}(f)\colon A \to \mathrm{Hom}_{\mathrm{Ab}}(B, C), \quad a \mapsto f(a, \text{-})\,.$$

Conversely, given a morphism of abelian groups $g\colon B \to \mathrm{Hom}_{\mathrm{Ab}}(A, C)$, we obtain a bilinear morphism

$$A \times B \to C, \quad g(a, b) = g(a)(b)\,.$$

These assignments are inverse to each other, showing that $\Phi^2_{A,C}$ is a bijection.

It remains to check that the bijection

$$\Phi_{A,C} := \Phi^2_{A,C} \circ \Phi^1_{A,C}\colon \ \mathrm{Hom}_{\mathrm{Ab}}(A \otimes B; C) \simeq \mathrm{Hom}_{\mathrm{Ab}}(A, \mathrm{Hom}_{\mathrm{Ab}}(B, C))$$

is natural in A and C. This amounts checking that for a morphism $A' \to A$ and a morphism $C \to C'$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Ab}}(A \otimes B; C) \arrow[r, "\Phi^1_{A,C}"] \arrow[d] & \mathrm{BiLin}(A, B; C) \arrow[r, "\Phi^2_{A,C}"] \arrow[d] & \mathrm{Hom}_{\mathrm{Ab}}(A, \mathrm{Hom}_{\mathrm{Ab}}(B, C)) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ab}}(A' \otimes B; C') \arrow[r, "\Phi^1_{A',C'}"] & \mathrm{BiLin}(A', B; C') \arrow[r, "\Phi^2_{A',C'}"] & \mathrm{Hom}_{\mathrm{Ab}}(A', \mathrm{Hom}_{\mathrm{Ab}}(B, C'))
\end{tikzcd}$$

This is readily checked on individual maps.

It thus remains to prove part (3). We give a proof sketch. We want to describe an isomorphism

$$\mathrm{Hom}_S(M \otimes_R X, Y) \simeq \mathrm{Hom}_R(X, \mathrm{Hom}_S(M, Y))\,,$$

natural in $X \in \mathrm{LMod}_R^{\mathrm{op}}$ and $Y \in \mathrm{LMod}_S$. We have $\mathrm{Hom}_S(M,Y) \subset \mathrm{Hom}_{\mathrm{Ab}}(M,Y)$, and thus

$$\mathrm{Hom}_R(X, \mathrm{Hom}_S(M,Y)) \subset \mathrm{Hom}_{\mathrm{Ab}}(X, \mathrm{Hom}_S(M,Y)) \subset \mathrm{Hom}_{\mathrm{Ab}}(X, \mathrm{Hom}_{\mathrm{Ab}}(M,Y))\,.$$

We also have inclusions

$$\mathrm{Hom}_S(M \otimes_R X, Y) \subset \mathrm{Hom}_S(M \otimes X, Y) \subset \mathrm{Hom}_{\mathrm{Ab}}(M \otimes X, Y)\,,$$

where the first inclusion is obtained by precomposition with the epimorphism $M \otimes X \to M \otimes_R X$. It is an inclusion by the universal property of the cokernel. The natural bijection Φ restricts to a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(M \otimes_R X, Y) \arrow[r, hook] \arrow[d, "\Phi"] & \mathrm{Hom}_S(M \otimes X, Y) \arrow[r, hook] \arrow[d, "\Phi"] & \mathrm{Hom}_{\mathrm{Ab}}(M \otimes X, Y) \arrow[d, "\Phi"] \\
\mathrm{Hom}_R(X, \mathrm{Hom}_S(M,Y)) \arrow[r, hook] & \mathrm{Hom}_{\mathrm{Ab}}(X, \mathrm{Hom}_S(M,Y)) \arrow[r, hook] & \mathrm{Hom}_{\mathrm{Ab}}(X, \mathrm{Hom}_{\mathrm{Ab}}(M,Y))
\end{tikzcd}$$

as one checks on individual maps. The left vertical natural bijection yields the desired adjunction. □

Corollary 1.6.37. *Let R, S be rings and M be an S-R-bimodule*

(1) The functor $M \otimes_R (\text{-})\colon \mathrm{LMod}_R \longleftrightarrow \mathrm{LMod}_S$ is right exact.

(2) Let $(N_j)_{j\in J} \subset \mathrm{LMod}_R$ be a family of R-modules. Then

$$M \otimes_R \left(\bigoplus_{j\in J} N_j\right) \simeq \bigoplus_{j\in J} (M \otimes_R N_j)\ .$$

Proof. That a left adjoint is right exact is shown on sheet 7. The second part (2) follows from left adjoints preserving colimits. □

Definition 1.6.38. let R, S be rings and M be an S-R-bimodule. Then M is called flat if $M \otimes_R (\text{-})$ is a left exact functor.

Note that an abelian group A is called flat if $A \otimes (\text{-})\colon \mathrm{Ab} \to \mathrm{Ab}$ is also left exact (since $\mathrm{Ab} \simeq \mathrm{LMod}_{\mathbb{Z}}$).

Example 1.6.39. Let $\phi\colon R \to S$ be a ring homomorphism. Then S is an S-R-bimodule, where the action of $r \in R$ on $s \in S$ is given by $s \cdot r = s \cdot \phi(r)$. The left S-action on S is given by multiplication in S. Then there is an adjunction

$$\phi_! \coloneqq S \otimes_R (\text{-})\, \mathrm{LMod}_R \longleftrightarrow \mathrm{LMod}_S\ : \mathrm{Hom}_S(S, \text{-}) =: \phi^*\,.$$

The functor ϕ^* is sometimes called the restriction of scalars.

Note that $\phi_!$ maps free R-modules to free S-modules. For $\phi\colon \mathbb{Z} \to \mathbb{Q}$, the functor $\phi_!$ maps $\mathbb{Z}$ to $\mathbb{Q}$ and

$$\phi_!(\mathbb{Z}/2\mathbb{Z}) \simeq \phi_!(\mathrm{coker}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})) \simeq \mathrm{coker}(\mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q}) \simeq 0\,,$$

since left adjoints preserve cokernels.

Proposition 1.6.40. *Let R be a ring and I, J be sets. We consider R as an R-R-bimodule. Then $R^{\oplus I} \otimes_R R^{\oplus J} \simeq R^{\oplus(I\times J)} \in \mathrm{LMod}_R$.*

Proof. Using part (2) of Corollary 1.6.37, and the symmetry of the considered tensor product, we can reduce to the case $I = J = *$. The functor $\mathrm{Hom}_R(R, \text{-})\colon \mathrm{LMod}_R \to \mathrm{LMod}_R$ is naturally isomorphic to the identity: given a left R-module M, we have $\mathrm{Hom}_R(R, M) \simeq M$, via the assignment $(f\colon R \to M) \mapsto f(1_R)$. By the uniqueness of adjoints, the left adjoint $R \otimes_R (\text{-})$ of $\mathrm{Hom}_R(R, \text{-})$ is therefore also naturally isomorphic to $\mathrm{id}_{\mathrm{LMod}_R}$. This shows that $R \otimes_R R \simeq R$. □

2 Derived functors

In this section, we will introduce *derived functors* of left or right exact functors.

We motivated the derived functor of the Hom functor in Section 0, see in particular Theorem 0.23: the derived functor allows to measure the failure of the exactness of the functor. In this chapter, we will systematically develop their theory along the following lines:

- We will introduce projective and injective objects in abelian categories. Additive functors preserve short exact sequences of projective/injective objects, since these split.
- We obtain the left derived functor of a right exact functor by applying the functor to projective resolutions. The arising collection of functors comes with a LES for each SES.
- We then study different examples of derived functors:
  - The derived functor of the Hom functor is called Ext^*. We will relate Ext^* with extensions.
  - The left derived functor of the tensor product functor is called Tor_* (because in the case $\mathcal{C} = \mathrm{Ab}$ it measures torsion).
  - The left derived functor of the fixed points functor for group actions (limit functor over BG) is called group cohomology.
- Finally, we introduce quivers and their representations, and exhibit different identities for Ext-groups of quiver representations.

2.1 Projective and injective objects

2.1.1 Definition and properties

Definition 2.1.1. Let $\mathcal{C}$ be an abelian category.

(1) An object P in $\mathcal{C}$ is called projective if for every epimorphism $p\colon B \to C$ and morphism $\gamma\colon P \to C$, there exists a morphism $\beta\colon P \to B$ with $p \circ \beta = \gamma$, meaning that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, two heads, "p"] & C \arrow[r] & 0 \end{tikzcd}$$

We also call β a lift of γ along p.

(2) An object I in $\mathcal{C}$ is called injective if for every monomorphism $i\colon A \to B$ and morphism $\alpha\colon A \to I$, there exists a morphism $\beta\colon A \to I$ with $\beta \circ i = \alpha$, meaning that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "i"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\exists\beta"] \\ & I & \end{tikzcd}$$

Proposition 2.1.2. *Let R be a ring. Then for an R-module $P \in \mathrm{LMod}_R$, the following are equivalent:*

(1) The module P is projective.

(2) The module P is a direct summand of a free R-module.

(3) The functor $\mathrm{Hom}_{\mathcal{C}}(P, \text{-})\colon \mathcal{C} \to \mathrm{Ab}$ is exact.

(4) Every short exact sequence in $\mathcal{C}$ of the form

$$0 \to A \to B \to P \to 0$$

splits.

Proof. (1) $\Longrightarrow$ (3): Suppose that P is projective. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in LMod_R. Since $\mathrm{Hom}_R(P, \text{-})$ is left exact, the sequence

$$0 \to \mathrm{Hom}_R(P, A) \xrightarrow{f \circ (\text{-})} \mathrm{Hom}_R(P, B) \xrightarrow{g \circ (\text{-})} \mathrm{Hom}_R(P, C)$$

is exact. To extend the above to a SES, we must show that $\mathrm{Hom}_R(P, B) \xrightarrow{g \circ (\text{-})} \mathrm{Hom}_R(P, C)$ is an epimorphism. This follows directly from the lifting property of the projective P, since g is an epimorphism.

(3) $\Longrightarrow$ (1): Suppose that $\mathrm{Hom}_R(P, \text{-})$ is exact and let $p\colon B \to C$ be an epimorphism. Then $\mathrm{Hom}_R(P, p)\colon \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is also an epimorphism by the exactness of $\mathrm{Hom}_R(P, \text{-})$. Thus any morphism $\gamma \in \mathrm{Hom}_R(P, C)$ admits a lift

$\beta \in \mathrm{Hom}_R(P, B)$ satisfying that $p \circ \beta = \mathrm{Hom}_R(P, p)(\beta) = \gamma$.

(1) $\implies$ (2): Let M be a projective left R-module. Then there exists an epimorphism $p\colon R^{\oplus M} \to M$ in LMod_R. The identity $\mathrm{id}_M\colon M \to M$ admits a lift along p to a morphism $\beta\colon M \to R^{\oplus M}$. Since $\beta \circ p = \mathrm{id}_M$, we see that β is a split monomorphism, showing that M is a direct summand of $R^{\oplus M}$.

(2) $\implies$ (3),(1): We first show that free left R-modules are projective. Let $R^{\oplus J}$ be a free left R-module with basis $\{\iota(j)\}_{j \in J}$, with $\iota\colon J \to R^{\oplus J}$ as in the proof of Proposition 1.1.10. Consider a morphism $\gamma\colon P \to C$ and an epimorphism $p\colon B \to C$. For each object $c_j = \gamma(\iota(j)) \in C$, $j \in J$, choose a preimage b_j under β. Then by Proposition 1.1.10, there exists a unique R-module homomorphism $\beta\colon R^{\oplus J} \to B$ with $\beta(\iota(j)) = b_j$. Then $p \circ \beta = \gamma$, as desired.

Let $M \subset R^{\oplus J}$ be a direct summand of a free left R-module with direct sum complement N. We now show that M is projective. Given a SES

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{18}$$

in LMod_R, since $M \oplus N$ is free, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M \oplus N, A) \xrightarrow{f \circ (\text{-})} \mathrm{Hom}_R(M \oplus N, B) \xrightarrow{g \circ (\text{-})} \mathrm{Hom}_R(M \oplus N, C) \to 0\,. \tag{19}$$

We note that $\mathrm{Hom}_R(M \oplus N, X) \simeq \mathrm{Hom}_R(M, X) \oplus \mathrm{Hom}_R(N, X)$ for all $X \in \mathcal{C}$. The above sequence is thus the direct sum of the two sequences

$$0 \to \mathrm{Hom}_R(M, A) \xrightarrow{f \circ (\text{-})} \mathrm{Hom}_R(M, B) \xrightarrow{g \circ (\text{-})} \mathrm{Hom}_R(M, C) \to 0$$

and

$$0 \to \mathrm{Hom}_R(N, A) \xrightarrow{f \circ (\text{-})} \mathrm{Hom}_R(N, B) \xrightarrow{g \circ (\text{-})} \mathrm{Hom}_R(N, C) \to 0$$

which must each be exact for the sequence (19) to be exact. We thus find that $\mathrm{Hom}_R(M, \text{-})$ is an exact functor.

(1) $\implies$ (4): Let $p\colon B \to P$ be an epimorphism. Then lifting id_P along p yields a section s of p. Thus, any short exact sequence $0 \to A \to B \to P \to 0$ splits.

(4) $\implies$ (2): Choose an epimorphism $p\colon R^{\oplus I} \to P$. Then the short exact sequence $0 \to \ker(p) \to R^{\oplus I} \to P \to 0$ splits, meaning that P is a direct summand of $R^{\oplus I}$. $\square$

Remark 2.1.3. The proof of Proposition 2.1.2 shows that in any abelian category $\mathcal{C}$, an object P is projective if and only if $\mathrm{Hom}_{\mathcal{C}}(P, \text{-})\colon \mathcal{C} \to \mathrm{Ab}$ is exact. It will follow from later results that this is also the case if and only if every short exact sequence $0 \to A \to B \to P \to 0$ in $\mathcal{C}$ splits. This means that (1) $\iff$ (3) $\iff$ (4) as in Proposition 2.1.2 holds in any abelian category.

Example 2.1.4. (1) An abelian group A is projective iff it is free and injective iff it is divisible, meaning that for each $x \in A$ and $n \geq 0$, there exists $b \in A$ with $b \cdot n = a$. Both these characterizations are non-trivial assertions.

(2) Let k be a field. Then every k-module is free and thus projective in LMod_k.

(3) Let $M_{n\times n}(k)$ be the ring of $n \times n$-matrices valued in the field k. Then $M_{n\times n}(k)$ is the direct sum $M_{n\times n}(k) = (k^{\oplus n})^{\oplus n}$ of n modules $k^{\oplus n}$ (column vectors), on which $M_{n\times n}(k)$ acts by left matrix multiplication. Thus, $k^{\oplus n} \in \mathrm{LMod}_{M_{n\times n}(k)}$ is projective but not free.
Considering $k^{\oplus n}$ as a $M_{n\times n}(k)$-k-bimodule (with the right k-action componentwise), we obtain a functor

$$k^{\oplus n} \otimes_k (\text{-})\colon\ \mathrm{LMod}_k \longrightarrow \mathrm{LMod}_{M_{n\times n}(k)}\ .$$

One can show that $k^{\oplus n} \otimes_k (\text{-})$ is an equivalence of categories. The property of being a projective object is preserved under such equivalences of categories, and so every $M_{n\times n}(k)$-module is projective. The property of being free is not preserved by the equivalence of categories.

Definition 2.1.5. Let $\mathcal{C}$ be an abelian category and $M \in \mathcal{C}$.

(1) A left resolution P_* of M consists of a chain complex P_* concentrated in positive degrees, together with an augmentation map $\epsilon\colon P_0 \to M$, such that the chain complex

$$\cdots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \to 0$$

is exact. We write $P_* \to M$ for left resolutions. If P_i is projective for all $i \geq 0$, we call $P_* \to M$ a projective resolution.

(2) A right resolution I^* of M consists of a cochain complex I^* concentrated in positive degrees, together with an augmentation map $\epsilon\colon M \to I^0$, such that the cochain complex

$$0 \to M \xrightarrow{\epsilon} I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \to \dots$$

is exact. We write $M \to I^*$ for right resolutions. If I^i is injective for all $i \geq 0$, we call $M \to I^*$ an injective resolution.

2.1.2 The Fundamental Lemma

Theorem 2.1.6 (Fundamental Lemma of Homological Algebra)**.** *Let $\mathcal{C}$ be an abelian category, $M, N \in \mathcal{C}$, and $f\colon M \to N$. Let further $\epsilon_M\colon P_* \to M$ be a projective resolution and $\epsilon_N\colon Q_* \to N$ a resolution (not necessarily projective).*

(1) There exists a chain map $f_*\colon P_* \to Q_*$ extending f, meaning that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r, "\epsilon_M"] \arrow[d, "f_0"] & M \arrow[d, "f"] \\
\dots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\epsilon_N"] & N
\end{tikzcd}$$

commutes.

(2) Any two chain maps $f_*, g_* \colon P_* \to Q_*$ extending f are chain homotopic.

Proof. Since $Q_* \to N$ is a resolution, the augmentation map $\epsilon_N \colon Q_0 \to N$ is an epimorphism. Since P_0 is projective, the morphism $f \circ \epsilon_M \colon P_0 \to M$ thus admits a lift, denoted f_0, along $\epsilon_N \colon Q_0 \to N$:

$$\begin{tikzcd}
P_0 \arrow[r, "\epsilon_M"] \arrow[d, dashed, "\exists f_0"'] \arrow[dr, "f_0 \circ \epsilon_M"] & M \arrow[d, "f"] & \\
Q_0 \arrow[r, two heads, "\epsilon_N"] & N \arrow[r] & 0
\end{tikzcd}$$

Using the universal property of the kernel, the kernels of ϵ_M, ϵ_N assemble into a commutative diagram as follows:

$$\begin{tikzcd}
P_1 \arrow[r, "(d_1^P)^{\mathrm{epi}}"'] \arrow[rr, bend left, "d_1^P"] & \mathrm{Im}(d_1^P) = \ker(\epsilon_M) \arrow[r, hook] \arrow[d, "f_0'"] & P_0 \arrow[r, "\epsilon_M"] \arrow[d, "f_0"] & M \arrow[d, "f"] \\
Q_1 \arrow[r, "(d_1^Q)^{\mathrm{epi}}"] \arrow[rr, bend right, "d_1^Q"'] & \mathrm{Im}(d_1^Q) = \ker(\epsilon_N) \arrow[r, hook] & Q_0 \arrow[r, "\epsilon_N"] & N
\end{tikzcd}$$

We also used the exactness of the resolutions to obtain the factorizations of d_1^P, d_1^Q through their images. We note that $(d_1^Q)^{\mathrm{epi}}$ is an epimorphism, so that we can lift $f_0' \circ (d_1^P)^{\mathrm{epi}}$ along $(d_1^Q)^{\mathrm{epi}}$ to a morphism $f_1 \colon P_1 \to Q_1$, satisfying that $d_1^Q \circ f_1 = f_0 \circ d_1^P$.

To construct the further morphisms $f_i \colon P_i \to Q_i$, $i \geq 2$, one proceeds as above, factorizing d_i^P, d_i^Q through their images and using the lifting property of the projective object P_i.

We turn to the proof of part (2). Let f_*, g_* be two chain maps extending f. Then $f_* - g_*$ extends $0 \colon M \to N$, and it suffices to show that $f_* - g_*$ is null homotopic. We thus assume that $f = 0$ and show that any chain map f_* constructed as above is nullhomotopic. For this, consider the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] \arrow[dl, dashed, "\exists H_0"'] \arrow[d, "f_0'"] \arrow[dr, "0"] & M \arrow[d, "0"] \\
{} \arrow[r] & Q_1 \arrow[r, two heads] \arrow[dr, "d_1^Q"'] & \mathrm{Im}(d_1^Q) = \ker(\epsilon_N) \arrow[r] \arrow[d, hook] & N \\
& & Q_0 \arrow[ur, "\epsilon_N"'] &
\end{tikzcd}$$

For this, we used the image factorization of d_1^Q, and the exactness of the resolution, to factor $f_0 \colon P_0 \to Q_0$ through $\mathrm{Im}(d_1^Q) = \ker(\epsilon_N)$ (calling the factorization f_0'). Since the morphism $Q_1 \to \mathrm{Im}(d_1^Q)$ is an epimorphism, and P_0 projective, we can lift f_0' along it, yielding the morphism $H_0 \colon P_0 \to Q_1$. It satisfies that $d_1^Q \circ H_0 = f_0$.

To proceed, we observe that

$$d_1^Q(f_1 - H_0 \circ d_1^P) = d_1^Q \circ f_1 - d_1^Q \circ H_0 \circ d_1^P = d_1^Q \circ f_1 - f_0 \circ d_1^P = 0\,,$$

since f_* is a chain map. We can thus reapply the above argument, using the factorization

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] \arrow[dl, dashed, "\exists H_1"'] \arrow[d, dashed, "(f_1 - H_0 \circ d_1^P)'"] \arrow[dr, "0"] & P_0 \\
{} \arrow[r] & Q_2 \arrow[r, two heads] \arrow[dr, "d_2^Q"'] & \mathrm{Im}(d_2^Q) = \ker(d_1^Q) \arrow[r] \arrow[d, hook] & Q_0 \\
 & & Q_1 \arrow[ur, "d_1^Q"'] &
\end{tikzcd}$$

of $f_1 - H_0 \circ d_1^P$. Then $d_2^Q \circ H_1 = f_1 - H_0 \circ d_1^P$, meaning that $f_1 = d_2^Q \circ H_1 + H_0 \circ d_1^P$. Iterating this argument yields the desired null homotopy H_* of f_*. □

Definition 2.1.7. Let $\mathcal{C}$ be an abelian category. We say that $\mathcal{C}$ has enough projectives if for every object M of $\mathcal{C}$ there exists a projective P and an epimorphism $p\colon P \to M$. We say that $\mathcal{C}$ has enough injectives if for every object M there exists a monomorphism $i\colon M \to I$ with I injective.

Example 2.1.8. For any ring R, the abelian category LMod_R has enough projectives. It also has enough injectives, but this is a non-trivial assertion.

Lemma 2.1.9. *Let $\mathcal{C}$ be an abelian category with enough projectives. Then every object of $\mathcal{C}$ admits a projective resolution. Dually, if $\mathcal{C}$ admits enough injectives, then every object of $\mathcal{C}$ admits an injective resolution.*

Proof. Let $M \in \mathcal{C}$. Then there exists an epimorphism $\epsilon\colon P_0 \to M$ with P_0 projective. We next choose an epimorphism $P_1 \to \ker(\epsilon)$ with P_1 projective and define $d_1\colon P_1 \to P_0$ as the composite $P_1 \to \ker(\epsilon) \to P_0$. We then iteratively define the projective P_i via a choice of epimorphism $P_i \to \ker(d_{i-1})$. By construction, the chain complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M$$

is exact. □

2.1.3 Enough injectives in LMod_R

The goal of this section is to prove the following:

Theorem 2.1.10. *Let R be a ring. Then LMod_R has enough injectives.*

Lemma 2.1.11. *Consider an adjunction $F\colon \mathcal{C} \leftrightarrow \mathcal{D} :G$ between abelian categories with F exact. Let $I \in \mathcal{D}$ be injective. Then $G(I) \in \mathcal{C}$ is also injective.*

Proof. We show that $\mathrm{Hom}_{\mathcal{C}}(\text{-}, G(I)) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is exact, which implies that $G(I)$ is projective in $\mathcal{C}^{\mathrm{op}}$, i.e. injective in $\mathcal{C}$. Since $\mathrm{Hom}_{\mathcal{C}}(\text{-}, G(I))$ is left exact, it suffices to show that that it preserves epimorphisms (meaning it maps epimorphisms in $\mathcal{C}^{\mathrm{op}}$ to epimorphisms in Ab). Let $f \colon A \to B$ be a monomorphism in $\mathcal{C}$. The bottom morphisms in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B, G(I)) \arrow[r, "(\text{-})\circ f"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(A, G(I)) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathcal{D}}(F(B), I) \arrow[r, "(\text{-})\circ F(f)"] & \mathrm{Hom}_{\mathcal{D}}(F(A), I)
\end{tikzcd}$$

is an epimorphism, since I is injective. Thus, so is also the top morphisms. $\square$

We next describe the adjunction to which we will apply the above lemma. Let R be a ring and let $F \colon \mathrm{LMod}_R \to \mathrm{Ab}$ be the forgetful functor. We established in Proposition 1.3.10 that the functor F is exact. Considering R as a $\mathbb{Z}$-R-module, and omitting the isomorphism of categories $\mathrm{Ab} \simeq \mathrm{LMod}_{\mathbb{Z}}$, we can identify F with the functor $R \otimes_R (\text{-})$. Its right adjoint is thus by Proposition 1.6.36.(3) given by $\mathrm{Hom}_{\mathrm{Ab}}(R, \text{-}) \colon \mathrm{Ab} \to \mathrm{LMod}_R$. Lemma 2.1.11 shows that $\mathrm{Hom}_{\mathrm{Ab}}(R, \text{-})$ maps injective abelian groups to injective R-modules.

Example 2.1.12. The abelian group $\mathbb{Q}/\mathbb{Z}$ is injective, since it is divisible. Thus $I_R \coloneqq \mathrm{Hom}_{\mathrm{Ab}}(R, \mathbb{Q}/\mathbb{Z}) \in \mathrm{LMod}_R$ is injective.

Lemma 2.1.13. *Let R be a ring and M a left R-module. Then the canonical morphism $i \colon M \to \prod_{\mathrm{Hom}_R(M, I_R)} I_R$ in* LMod_R *is a monomorphism.*

Proof. We fix $0 \neq m \in M$ and show that there is an R-module homomorphism $g \colon M \to I_R$ such that $g(m) \neq 0$. This shows that the kernel of i is trivial, meaning that i is a monomorphism.

We first consider the abelian subgroup $\langle m \rangle$ of M generated by m (this subgroup is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some $n > 1$). We define the non-zero morphism $\alpha \colon \langle m \rangle \to \mathbb{Q}/\mathbb{Z}$ by $\alpha(\phi(1)) = \frac{1}{2}$ if $\phi \colon \mathbb{Z} \simeq \langle m \rangle$ and by $\alpha(\phi(1)) = \frac{1}{n}$ if $\phi \colon \mathbb{Z}/n\mathbb{Z} \simeq \langle m \rangle$.

Using that $\mathbb{Q}/\mathbb{Z}$ is injective, we can find a lift β as follows:

$$\begin{tikzcd}
\langle m \rangle \arrow[r, hook] \arrow[d, "\alpha"'] & M \arrow[dl, dashed, "\beta"] \\
\mathbb{Q}/\mathbb{Z} &
\end{tikzcd}$$

We define $g \colon M \to I_R$ by $n \mapsto (r \mapsto \beta(r \cdot n))$. One readily checks that g is an R-module homomorphism. We find that $g(m)(1_R) = \beta(m) \neq 0$, so $g(m) \neq 0$. $\square$

Proof of Theorem 2.1.10. Let M be a left R-module. We note that any product of injective objects is injective. Thus $\prod_{\mathrm{Hom}_R(M, I_R)} I_R$ is injective. The monomorphism i from Lemma 2.1.13 thus shows that LMod_R has enough injectives. $\square$

2.2 Derived functors

2.2.1 Definition and the LES

Definition 2.2.1. Let $\mathcal{C}, \mathcal{D}$ be abelian categories.

(1) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a right exact functor and assume that $\mathcal{C}$ has enough projectives. Then for $M \in \mathcal{D}$ and $n \geq 0$, define $L_nF(M) \in \mathcal{D}$ as

$$L_nF(M) \coloneqq H_n(F(P_*))$$

with $P_* \to M$ a projective resolution.

(2) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left exact functor and assume that $\mathcal{C}$ has enough injectives. Then for $M \in \mathcal{D}$ and $n \geq 0$, define $R^nF(M) \in \mathcal{D}$ as

$$R^nF(M) \coloneqq H^n(F(I^*))$$

with $M \to I^*$ an injective resolution.

The above definition is well defined:

Proposition 2.2.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be right exact and assume that $\mathcal{C}$ has enough projectives.*

(1) For any $M \in \mathcal{C}$, the object $L_nF(M)$ is independent on the choice of projective resolution $P_* \to M$, up to isomorphism in $\mathcal{C}$.

(2) The assignment $M \mapsto L_nF(M)$ defines a functor $L_nF(\text{-})\colon \mathcal{C} \to \mathcal{D}$, called the n-th left derived functor of F.

(3) There exists a natural isomorphism $L_0F \simeq F\colon \mathcal{C} \to \mathcal{D}$.

If F is instead left exact and $\mathcal{C}$ has enough injectives, similar assertions hold for the n-th right derived functor $R^nF\colon \mathcal{C} \to D$.

Proof. Let $M \in \mathcal{C}$ and consider two projective resolutions $P_*, Q_* \to M$. Applying Theorem 2.1.6.(1) twice, we can lift the identity $\mathrm{id}_M\colon M \to M$ to a chain maps $f_*\colon P_* \to Q_*$ and $g_*\colon Q_* \to P_*$. We note that $g_* \circ f_*\colon P_* \to P_*$ extends the identity on M, as does id_{P_*}. By Theorem 2.1.6.(2), we find that g_*f_* is chain homotopic to id_{P_*}. A similar argument shows that f_*g_* is chain homotopic to id_{Q_*}. Thus, f_* is a chain homotopy equivalence. Using that additive functors preserve chain homotopy equivalences (by applying the functors to the involved homotopies), we find that the chain map $F(f_*)\colon F(P_*) \to F(Q_*)$ is thus again a chain homotopy equivalence. It follows that $H_i(F(f_*))\colon H_i(F(P_*)) \to H_i(F(Q_*))$ is an isomorphism, showing part (1).

We turn to the construction of the functor $L_nF(\text{-})$. Let $f\colon M \to N$ be a morphism in $\mathcal{C}$. Choosing two projective resolutions $P_* \to M$ and $Q_* \to N$, we can lift f to a chain map $f_*\colon P_* \to Q_*$ by Theorem 2.1.6.(1). We define

$L_nF(f) = H_n(f_*)\colon L_nF(M) = H_n(P_*) \to L_nF(N) = H_n(Q_*)$. This is well defined: a different choice of lift g_* of f is chain homotopic to f_* by Theorem 2.1.6, and thus $H_n(f_*) = H_n(g_*)$ by Lemma 1.5.19.

This defines a functor: We first note that $L_nF(\mathrm{id}_M) = \mathrm{id}_{L_nF(M)}$, since id_{P_*} is a lift of id_M. Next, consider two morphisms $f\colon M \to M'$ and $g\colon M' \to M''$ in $\mathcal{C}$ and projective resolutions $P_* \to M$, $P'_* \to M'$, $P''_* \to M''$. Consider two lifts of f, g to chain maps $f_*\colon P_* \to P'_*$ and $g_*\colon P'_* \to P''_*$. Then $g_* \circ f_*$ is a lift of $g \circ f$, and thus $L_nF(g \circ f) = H_n(g_* \circ f_*) = H_n(g_*) \circ H_n(f_*) = L_nF(g) \circ L_nF(f)$, since $H_n(\text{-})$ is a functor. This concludes the proof of part (2).

We turn to proof of part (3). Let $\epsilon\colon P_* \to M$ be a projective resolution. Applying the right exact functor F to the exact sequence

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

yields an exact sequence

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(\epsilon)} F(M) \to 0\,.$$

This induces an isomorphism $\eta_M\colon L_0F(M) = F(P_0)/\operatorname{Im}(F(d_1)) \simeq F(M)$. These isomorphisms assemble into a natural transformation $\eta = (\eta_x)_{x\in\mathcal{C}}$: Given a morphism $f\colon M \to N$, we lift f to a chain map $f_*\colon P_* \to Q_*$ between projective resolutions of M, N. The commutativity of the diagram

$$\begin{array}{ccccc}
F(P_1) & \xrightarrow{F(d_1^P)} & F(P_0) & \longrightarrow & F(M) \\
\big\downarrow{\scriptstyle F(f_1)} & & \big\downarrow{\scriptstyle F(f_0)} & & \big\downarrow{\scriptstyle F(f)} \\
F(Q_1) & \xrightarrow{F(d_1^Q)} & F(Q_0) & \longrightarrow & F(N)
\end{array}$$

implies the commutativity of the arising square

$$\begin{array}{ccc}
F(P_0)/\operatorname{Im}(F(d_1^P)) & \xrightarrow{\simeq} & F(M) \\
{\scriptstyle H_0(f_*)=L_0F(f)}\big\downarrow & & \big\downarrow{\scriptstyle F(f)} \\
F(Q_0)/\operatorname{Im}(F(d_1^Q)) & \xrightarrow{\simeq} & F(N)
\end{array}$$

showing the desired naturality. □

Remark 2.2.3.

(1) If F is an exact functor, then $L_nF \simeq 0$ and $R^nF \simeq 0$ for all $n > 0$.

(2) If $P \in \mathcal{C}$ is projective, then

$$L_nF(P) \simeq \begin{cases} F(P) & n = 0 \\ 0 & n > 0 \end{cases}$$

for any right exact functor F.

Theorem 2.2.4. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories.*

(i) Let $F\colon \mathcal{C} \to \mathcal{D}$ be right exact and assume that $\mathcal{C}$ has enough projectives. Then for any SES in $\mathcal{C}$
$$0 \to A \to B \to C \to 0 \tag{20}$$
there exists a LES in $\mathcal{D}$:
$$\cdots \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0 \tag{21}$$
Assinging to the SES (20) in $\mathcal{C}$ the LES (21) in $\mathcal{D}$ defines a functor $\mathrm{SES}(\mathcal{C}) \to \mathrm{LES}(\mathcal{D})$. *This fact is also referred to as the naturality of the LES.*

(ii) Let $F\colon \mathcal{C} \to \mathcal{D}$ be left exact and assume that $\mathcal{C}$ has enough injectives. Then for any SES in $\mathcal{C}$
$$0 \to A \to B \to C \to 0$$
there exists a LES in $\mathcal{D}$:
$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots$$
Similar to above, the assignment of this LES defines a functor $\mathrm{SES}(\mathcal{C}) \to \mathrm{LES}(\mathcal{D})$.

Lemma 2.2.5 (Horseshoe lemma)**.** *Let $\mathcal{C}$ be an abelian category and $0 \to M \to M' \to M'' \to 0$ a SES in $\mathcal{C}$. Let $\epsilon\colon P_* \to M$ and $\epsilon''\colon P''_* \to M''$ be projective resolutions. Then there exists a projective resolution $\epsilon'\colon P'_* \to M'$ fitting into a SES in* $\mathrm{Ch}(\mathcal{C})$
$$0 \to P_* \to P'_* \to P''_* \to 0$$
extending the SES $0 \to M \to M' \to M'' \to 0$, meaning that the diagram
$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
P_1 \arrow[r] \arrow[d] & P'_1 \arrow[r] \arrow[d] & P''_1 \arrow[d] \\
P_0 \arrow[r] \arrow[d, "\epsilon"] & P'_0 \arrow[r] \arrow[d, "\epsilon'"] & P''_0 \arrow[d, "\epsilon''"] \\
M \arrow[r] & M' \arrow[r] & M''
\end{tikzcd}$$
commutes.

Proof. We set $P'_i = P_i \oplus P''_i$ for all $i \geq 0$ and must assemble these into a chain complex fitting into a SES $0 \to P_* \to P'_* \to P''_*$ in $\mathrm{Ch}(\mathcal{C})$. In each degree, the morphisms $P_* \to P'_*$ and $P'_* \to P''_*$ will be given by the apparent inclusion and projection.

Using that $M' \to M''$ is an epimorphism, we can find a lift $\epsilon_2 \colon P_0'' \to M'$ of ϵ''. We define $\epsilon_1 \colon P_0 \to M'$ as the composite of $\epsilon \colon P_0 \to M$ with $M \to M'$. We set $\epsilon' = (\epsilon_1, \epsilon_2) \colon P_0 \oplus P_0'' \to M'$. We can illustrate these definitions via the following diagram (note that one of the triangles does not commute in the diagram!):

$$\begin{tikzcd}
P_0 \arrow[r, hook] \arrow[d, "\epsilon"] \arrow[dr, "\epsilon_1"] & P_0 \oplus P_0'' \arrow[r, two heads] \arrow[d, "(\epsilon_1,\epsilon_2)"] & P_0'' \arrow[d, "\epsilon''"] \arrow[dl, dashed, "\epsilon_2"'] \\
M \arrow[r] & M' \arrow[r] & M''
\end{tikzcd}$$

Applying the snake lemma, and using that $\operatorname{coker}(\epsilon) = \operatorname{coker}(\epsilon'') = 0$, we find that $\operatorname{coker}(\epsilon') = 0$, meaning that ϵ' is an epimorphism. The snake lemma further supplies us with the middle exact sequence in the following diagram:

$$\begin{tikzcd}
P_1 \arrow[r, hook] \arrow[d] \arrow[dr, "\tilde{d}_1"'] & P_1 \oplus P_1'' \arrow[r, two heads] \arrow[d, dashed, "\tilde{d}_1'=(\tilde{d}_1,\tilde{d}_1'')"] & P_1'' \arrow[d] \arrow[dl, dashed, "\tilde{d}_1''"] \\
\ker(\epsilon) \arrow[r] \arrow[d] & \ker(\epsilon') \arrow[r] \arrow[d, "\iota'"] & \ker(\epsilon'') \arrow[d] \\
P_0 \arrow[r, hook] & P_0 \oplus P_0'' \arrow[r, two heads] & P_0''
\end{tikzcd}$$

We define a morphism $\tilde{d}_1' \colon P_1' = P_1 \oplus P_1'' \to \ker(\epsilon')$ using a lift as above, and set the first differential of P_*' to be $d_1' = \iota' \circ \tilde{d}_1'$. Iterating this construction yields the desired projective resolution $P_*' \to M'$. $\square$

Proof of Theorem 2.2.4. We only prove part (1), the proof of part (2) is dual. Let $0 \to A \to B \to C \to 0$ be a SES in $\mathcal{C}$ and $\epsilon \colon P_* \to A$ and $\epsilon'' \colon P_*'' \to C$ projective resolutions. By the Horseshoe Lemma 2.2.5, we can find a projective resolution $\epsilon' \colon P_*' \to B$ together with a short exact sequence of chain complexes $0 \to P_* \to P_*' \to P_*'' \to 0$. Applying F, and using that F preserves split exact sequences, see Remark 1.5.15, we obtain the short exact sequence of chain complexes $0 \to F(P_*) \to F(P_*') \to F(P_*'') \to 0$. The desired LES now arises from Theorem 1.5.21. We note that the LES is independent on the choice of projective resolution, up to isomorphism of long exact sequences, by the Fundamental Lemma Theorem 2.1.6.

Given a morphism between short exact sequences, meaning a commutative diagram with short exact rows as follows,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Theorem 2.1.6 allows us obtain a morphism between the chosen SES of projective resolutions extending the morphism between short exact sequences. The desired morphism between the LES thus arises from applying the functor from Theorem 1.5.21. $\square$

Definition 2.2.6.

(1) Let $\mathcal{C}$ be an abelian category with enough projectives. Given two objects $A, B \in \mathcal{C}$, we define the n-th extension group between A and B in $\mathcal{C}$ as

$$\mathrm{Ext}^n_{\mathcal{C}}(A,B) \coloneqq R^n \operatorname{Hom}_{\mathcal{C}}(\text{-}, B)(A) \in \mathrm{Ab}\,.$$

(2) Let R be a ring and $M \in \mathrm{RMod}_R$ and $N \in \mathrm{LMod}_R$. Then we define

$$\mathrm{Tor}^R_n(M,N) \coloneqq L_n(M \otimes_R \text{-})(N) \in \mathrm{Ab}\,.$$

Example 2.2.7. Let p be a prime number. Consider the two $\mathbb{Z}$-modules $M = N = \mathbb{Z}/p$ and consider the projective resolution $P_* \to B$ corresponding to the exact chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = N\,.$$

Then $\mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z}/p, \mathbb{Z}/p)$ is the n-th homology of the chain complex

$$\cdots \to 0 \to \mathbb{Z}/p \otimes \mathbb{Z} \xrightarrow{\mathrm{id}_{\mathbb{Z}/p} \otimes (\cdot p)} \mathbb{Z}/p \otimes \mathbb{Z}$$

which is isomorphic to the chain complex

$$\cdots \to 0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z}\,.$$

Thus

$$\mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z}/p, \mathbb{Z}/p) \simeq \begin{cases} \mathbb{Z}/p & n = 0, 1 \\ 0 & n \geq 2\,. \end{cases}$$

Remark 2.2.8. For completeness, we note that $\mathrm{Tor}^R_n(M,N)$ can also be computed using a flat resolution of N (instead of a more restrictive projective resolution). See also Lemma 3.2.8 in Weibel.

2.2.2 Ext and extensions

Let $\mathcal{C}$ be an abelian category and $A, B \in \mathcal{C}$. Recall from Definition 1.5.13 that we denote by

$$\mathrm{Ext}_{\mathcal{C}}(A,B) = \{B \to E \to A \text{ extension in } \mathcal{C}\}/\sim$$

the set of equivalence classes of extensions of A by B in $\mathcal{C}$.

Suppose that $\mathcal{C}$ has enough injectives and consider an extension $\xi = [0 \to B \to E \to A \to 0]$ in $\mathcal{C}$. Applying the functor $\operatorname{Hom}_{\mathcal{C}}(\text{-}, B)$ yields the LES

$$0 \to \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{C}}(E,B) \to \operatorname{Hom}_{\mathcal{C}}(B,B) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{C}}(A,B) \to \dots$$

Lemma 2.2.9. *The assignment*

$$\Theta\colon\ \mathrm{Ext}_{\mathcal{C}}(A,B) \to \mathrm{Ext}^1_{\mathcal{C}}(A,B), \quad [\xi] \mapsto \partial(\mathrm{id}_B)$$

is well defined.

Proof. An equivalence of extensions

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

defines an isomorphism in $\mathrm{SES}(\mathcal{C})$ and thus induces by Theorem 2.2.4 an isomorphism in LES(Ab) between the associated LES:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}_{\mathcal{C}}(E,B) \arrow[r] \arrow[d, "(\text{-})\circ\alpha"] & \mathrm{Hom}_{\mathcal{C}}(B,B) \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"] & \mathrm{Ext}^1_{\mathcal{C}}(A,B) \arrow[r] \arrow[d, "\mathrm{id}"] & \dots \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(E',B) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(B,B) \arrow[r, "\partial"] & \mathrm{Ext}^1_{\mathcal{C}}(A,B) \arrow[r] & \dots
\end{tikzcd}$$

□

Theorem 2.2.10. *Let $\mathcal{C}$ be an abelian category with enough projectives. Given two objects $A, B \in \mathcal{C}$, the morphism*

$$\Theta\colon \mathrm{Ext}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(A,B)$$

is a bijection.

Proof. Choose a projective resolution $\epsilon\colon P_* \to A$ and consider the arising SES $P_1/\operatorname{Im}(d_2) \to P_0 \xrightarrow{\epsilon} A$. Then there is a LES

$$\mathrm{Hom}_{\mathcal{C}}(P_0,B) \to \mathrm{Hom}_{\mathcal{C}}(P_1/\operatorname{Im}(d_2),B) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{C}}(A,B) \to \underbrace{\mathrm{Ext}^1_{\mathcal{C}}(P_0,B)}_{\simeq 0} \to \dots$$

where the right term vanishes since P_0 is projective. The morphism ∂ is thus an epimorphism. Given $[\alpha] \in \mathrm{Ext}^1_{\mathcal{C}}(A,B)$, we can thus find a morphism $\alpha\colon P_1/\operatorname{Im}(d_2) \to B$ with

$$\partial(\alpha) = [\alpha]\,. \tag{22}$$

Passing to a pushout, we find the following commutative diagram

$$\begin{tikzcd}
P_1/\operatorname{Im}(d_2) \arrow[r, hook] \arrow[d, "\alpha"] \arrow[dr, phantom, "\ulcorner"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "\beta"] & A \arrow[d, "\mathrm{id}_A"] \\
B \arrow[r, "f"] \arrow[rr, bend right, "0"'] & X \arrow[r, dashed, "g"] & A
\end{tikzcd} \tag{23}$$

where the dashed arrow is induced by the universal property of the pushout. Since pushouts of monomorphisms are monomorphisms (see Exercise 4 on Sheet 4), the morphism f is a monomorphism. Since $P_0 \to A$ is an epimorphism and factors through g, the morphism g must also be an epimorphism. We note that $X =$

$\mathrm{coker}(P_1/\mathrm{Im}(d_2) \to P_0 \oplus B)$. Let $(p,b) \in P_0 \oplus B$ whose image $[(p,b)] \in X$ lies in the kernel of g. Then $\epsilon(p) = g([(p,b)]) = 0$, and thus $p = d_1(p')$ for some $p' \in P_1/\mathrm{Im}(d_2)$. Thus $[(p,b)] = [(p - \alpha(p'), 0)] = f(p - \alpha(p')) \in X$ lies in the image of f.

We thus see that $B \xrightarrow{f} X \xrightarrow{g} A$ is a short exact sequence, and we define $\Psi([\alpha]) \in \mathrm{Ext}_{\mathcal{C}}(A,B)$ as this extension of A by B. It remains to show three things:

1) $\Psi([\alpha])$ is well defined, meaning independent of the chosen representative α.
2) $\Theta \circ \Psi([\alpha]) = [\alpha]$.
3) $\Psi \circ \Theta(e) = e$ for each equivalence class of extensions $e = [(B \to E \to A)]$.

1): Let $\gamma\colon P_0 \to B$. We must show that replacing α by $\alpha + \gamma \circ d_1$ yields an equivalent extension $\Psi([\alpha])$. Let Y be the following pushout:

$$\begin{tikzcd} P_1/\mathrm{Im}(d_2) \arrow[r, hook] \arrow[d, "\alpha+\gamma\circ d_1"'] \arrow[dr, phantom, "\ulcorner"] & P_0 \arrow[d] \\ B \arrow[r, "f'"] & Y \end{tikzcd}$$

A small computation shows that the morphism $P_0 \oplus B \xrightarrow{(\beta + f\circ\gamma, f)} X$ induces a morphism $h\colon Y \to X$ which extends to an equivalence of short exact sequences:

$$\begin{tikzcd} B \arrow[r, "f'"] \arrow[d, "\mathrm{id}_B"'] & Y \arrow[r] \arrow[d, "h"] & A \arrow[d, "\mathrm{id}_A"] \\ B \arrow[r, "f"] & X \arrow[r] & A \end{tikzcd}$$

2): We can consider the diagram (23) as a morphism between short exact sequences. We thus obtain an associated morphism between the LES, containing the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,B) \arrow[r, "\partial"] \arrow[d, "(\text{-})\circ\alpha"'] & \mathrm{Ext}^1_{\mathcal{C}}(A,B) \arrow[d, "\mathrm{id}"] \\ \mathrm{Hom}_{\mathcal{C}}(P_1/\mathrm{Im}(d), B) \arrow[r, "\partial"] & \mathrm{Ext}^1_{\mathcal{C}}(A,B) \end{tikzcd} \tag{24}$$

We thus see that

$$\Theta \circ \Psi([\alpha]) \overset{\text{def. }\Theta}{=} \partial(\mathrm{id}_B) \overset{(24)}{=} \partial(\alpha) \overset{(22)}{=} [\alpha]\,.$$

3): Fix an extension $e = (B \to E \to A)$. We can consider $\cdots \to 0 \to B \to E$ as a resolution of A. By the Fundamental Lemma Theorem 2.1.6, we obtain a morphism between resolution

$$\begin{tikzcd} \dots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{id}_A"] \\ \dots \arrow[r] & 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \end{tikzcd}$$

which induces a morphism between short exact sequences

$$\begin{tikzcd} P_1/\operatorname{Im}(d_2) \arrow[r] \arrow[d, "\alpha"] & P_0 \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{id}_A"] \\ B \arrow[r] & E \arrow[r] & A \end{tikzcd}$$

We note that $\partial(\alpha) = \Theta(e)$, as follows from the uniqueness of of the morphisms between resolutions up to chain homotopy in Theorem 2.1.6. Passing to the pushout $X = B \amalg_{P_1/\operatorname{Im}(d_2)} P_0$, we obtain a morphism between short exact sequences

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\mathrm{id}_B"'] & X \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{id}_A"] \\ B \arrow[r] & E \arrow[r] & A \end{tikzcd}$$

which is an equivalence of short exact sequences (by definition of these). The upper short exact sequence describes a representative of $\Psi \circ \Theta(e)$, so $\Psi \circ \Theta(e) = e \in \operatorname{Ext}_{\mathcal{C}}(A, B)$, as desired.

□

Remark 2.2.11. Inspecting the proof, we see that under the bijection Θ from Theorem 2.2.10, the split extension corresponds to the null element $0 \in \operatorname{Ext}^1_{\mathcal{C}}(A, B)$.

A consequence of Theorem 2.2.10 is that the set of extensions $\operatorname{Ext}_{\mathcal{C}}(A, B)$ inherits the structure of an abelian group. We next describe this sum, which is called the *Baer sum*.

Lemma 2.2.12. *Let $\mathcal{C}$ be an abelian category.*

(1) Consider a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[r] \arrow[d] & C \arrow[d, "\mathrm{id}_C"] \\ D \arrow[r] & E \arrow[r] & C \end{tikzcd}$$

whose rows are chain complexes and where the left square is pushout. If the top row is a short exact sequence, then so is the bottom row.

(2) Consider a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

whose rows are chain complexes and where the right square is pullback. If the bottom row is a short exact sequence, then so is the top row.

Proof. This follows from the argument given below the diagram (23). $\square$

Construction 2.2.13 (Baer sum, 1934)**.** Let $\mathcal{C}$ be an abelian category[3] and let $0 \to B \xrightarrow{f_1} E_1 \xrightarrow{g_1} A \to 0$ and $0 \to B \xrightarrow{f_2} E_2 \xrightarrow{g_2} A \to 0$ be two extensions in $\mathcal{C}$. Then applying Lemma 2.2.12 twice, we find that the top row in the following diagram is an extension of A by B.

$$\begin{tikzcd} B \arrow[r, "f"] & E \arrow[r, "g"] & A \\ B \oplus B \arrow[u, "(\mathrm{id}_B,\mathrm{id}_B)"] \arrow[r] \arrow[d, "\mathrm{id}_{B\oplus B}"'] \arrow[ur, phantom, "\llcorner"] & \tilde{E} \arrow[u, "p"] \arrow[r] \arrow[d, "i"] \arrow[dr, phantom, "\lrcorner"] & A \arrow[u, "\mathrm{id}_A"'] \arrow[d, "(\mathrm{id}_A,\mathrm{id}_A)"] \\ B \oplus B \arrow[r, "(f_1,f_2)"'] & E_1 \oplus E_2 \arrow[r, "(g_1,g_2)"'] & A \oplus A \end{tikzcd} \tag{25}$$

We call the extension

$$(B \to E \to A) = (B \to E_1 \to A) +_B (B \to E_2 \to A)$$

the Baer sum of $(B \to E_1 \to A)$ and $(B \to E_2 \to A)$.

The *pasting law* for pullback squares states in a diagram consisting of two squares, such as the diagram

$$\begin{tikzcd} \tilde{E} \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r] \arrow[d, dashed, "(\mathrm{id}_A,\mathrm{id}_A)"] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ E_1 \oplus E_2 \arrow[r, "(g_1,g_2)"'] & A \oplus A \arrow[r, "(\mathrm{id}_A,-\mathrm{id}_A)"'] & A \end{tikzcd}$$

if by counting the outer square as the third square, any two of the three squares are pullback, then so is the third. We thus find that the outer square in the above square is pullback. Similarly, there is a pushout square:

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \tilde{E} \arrow[d] \\ 0 \arrow[r] & E \end{tikzcd}$$

Proposition 2.2.14. *Let $\mathcal{C}$ be an abelian category with enough projectives. Consider two extensions $e_1 = (B \xrightarrow{f_1} E_1 \xrightarrow{g_1} A)$ and $e_2 = (B \xrightarrow{f_2} E_2 \xrightarrow{g_2} A)$ in $\mathcal{C}$. Then*

$$\Theta(e_1 +_B e_2) = \Theta(e_1) + \Theta(e_2) \in \mathrm{Ext}^1_{\mathcal{C}}(A, B)\,.$$

Proof. Let $e_1 +_B e_2 = (B \xrightarrow{f} E \xrightarrow{g} A)$. Choosing a projective resolution $P_* \to A$, and considering the corresponding commutative diagrams

$$\begin{tikzcd} P_1/\operatorname{Im}(d_2) \arrow[r] \arrow[d, "\alpha_i"'] & P_0 \arrow[r] \arrow[d, "\beta_i"] & A \arrow[d, "\mathrm{id}_A"] \\ B \arrow[r, "f_i"'] & E_i \arrow[r, "g_i"'] & A \end{tikzcd}$$

[3]We do not need to assume that $\mathcal{C}$ has enough projectives for the Baer sum to exist.

for $i = 1, 2$, we must show that there is a commutative diagram of the following form:

$$\begin{tikzcd}
P_1/\operatorname{Im}(d_2) \arrow[r] \arrow[d, "\alpha_1+\alpha_2"'] & P_0 \arrow[r] \arrow[d, "\beta"] & A \arrow[d, "\mathrm{id}_A"] \\
B \arrow[r, "f"'] & E \arrow[r, "g"'] & A
\end{tikzcd} \tag{26}$$

For that, consider the following commutative diagram:

$$\begin{tikzcd}
P_1/\operatorname{Im}(d_2) \arrow[r] \arrow[d, "{(\mathrm{id},\mathrm{id})}"] \arrow[dd, bend right=60, "\alpha_1+\alpha_2"'] & P_0 \arrow[r, dotted, "0"] \arrow[d, "{(\mathrm{id},\mathrm{id})}"'] & A & \\
(P_1/\operatorname{Im}(d_2))^{\times 2} \arrow[r] \arrow[d, "{(\gamma_1,\gamma_2)}"] & P_0^{\times 2} \arrow[r, "{(\beta_1,\beta_2)}"] & E_1 \times E_2 \arrow[u, "{(g_1,-g_2)}"'] \arrow[r, "{(g_1,0)}"] & A \\
B \arrow[rr] \arrow[rrr, bend right=15, "f"'] & & \tilde{E} \arrow[u, "i"] \arrow[r, "p"] & E \arrow[u, "g"']
\end{tikzcd}$$

The morphism $\tilde{f}$ is given by either component of the morphism $B \oplus B \to \tilde{E}$ in the diagram (25). Since the dotted arrow vanishes, and E is the kernel of $(g_1, -g_2)$, there is an induced morphism $\tilde{\beta}\colon P_0 \to \tilde{E}$, and we declare $\beta = p \circ \tilde{\beta}$. Inspecting the above diagram, we find that the diagram (26) commutes. $\square$

Example 2.2.15. Let p be a prime number. Then $\operatorname{Ext}^1_{\mathrm{Ab}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$. There are thus p equivalence classes of extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$. These are the split extension, as well as the $p-1$ extensions

$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\cdot x} \mathbb{Z}/p\mathbb{Z}$$

with $x = 1, \ldots, p-1$.

Remark 2.2.16. The n-th extension groups in an abelian category can be described in terms of exact sequences of length $n+2$. This decription of $\operatorname{Ext}^n_{\mathcal{C}}(A, B)$ is due to Yoneda and called *Yoneda Ext*. For $\operatorname{Ext}^n_{\mathcal{C}}(A, B)$, one considers exact sequences of the form

$$e = (0 \to B \to X_n \to X_{n-1} \to \cdots \to X_1 \to A \to 0)$$

modulo the equivalence relation arising from identifying two exact sequences in a commutative diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\mathrm{id}_B"'] & X_n \arrow[r] \arrow[d] & \ldots \arrow[r] & X_1 \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{id}_A"] \\
B \arrow[r] & Y_n \arrow[r] & \ldots \arrow[r] & Y_1 \arrow[r] & A
\end{tikzcd}$$

For $n \geq 2$, the sum of two Yoneda extensions is the Yoneda extension

$$B \to X_n \amalg_B Y_n \to X_{n-1} \oplus Y_{n-1} \to \cdots \to X_2 \oplus Y_2 \to X_1 \times_A Y_1 \to A\,.$$

Note that the existence of projective resolutions is not necessary to define the Yoneda-Ext, so these allow to define Ext^n in arbitrary abelian categories.

2.2.3 Balancing Ext^* and Tor_*

The goal of this section is to show that we can compute $\mathrm{Ext}^n_{\mathcal{C}}(A, B)$ using an injective resolution of A, instead of a projective resolution of B, and similarly, that we can compute $\mathrm{Tor}^n_R(M, N)$ using a projective resolution of N instead of M.

Definition 2.2.17. Let $\mathcal{C}$ be an additive category. A double chain complex $C_{*,*}$ in $\mathcal{C}$ consists of a $\mathbb{Z} \times \mathbb{Z}$-family of objects $(C_{i,j})_{i,j \in \mathbb{Z}\times\mathbb{Z}}$ together with two differentials $d^h_{i,j} \colon C_{i,j} \to C_{i-1,j}$ (horizontal) and $d^v_{i,j} \colon C_{i,j} \to C_{i,j-1}$ (vertical) for all $i, j \in \mathbb{Z}$, satisfying that:

$$\begin{aligned} (d^v)^2 &= 0 \\ (d^h)^2 &= 0 \\ d^h \circ d^v + d^v \circ d^h &= 0\,. \end{aligned}$$

A double chain complex is called bounded if for all $p \in \mathbb{Z}$ there are only finitely many non-zero terms of the form $C_{i,j}$ with $i + j = p$.

We can depict a double complex as follows, where each square anti-commutes:

$$\begin{tikzcd}
C_{i+1,j+1} \arrow[r, "d^h"'] \arrow[d, "d^v"] & C_{i,j+1} \arrow[r, "d^h"'] \arrow[d, "d^v"] & C_{i-1,j+1} \arrow[d, "d^v"] \\
C_{i+1,j} \arrow[r, "d^h"] \arrow[d, "d^v"] & C_{i,j} \arrow[r, "d^h"] \arrow[d, "d^v"] & C_{i-1,j} \arrow[d, "d^v"] \\
C_{i+1,j-1} \arrow[r, "d^h"] & C_{i,j-1} \arrow[r, "d^h"] & C_{i-1,j-1}
\end{tikzcd}$$

Note that a double complex does not quite define an object in $\mathrm{Ch}(\mathrm{Ch}(\mathcal{C}))$, since this would require each square to commute. This can however be achieved by replacing d^v with $(-1)^i d^v_{i,j}$.

Double cochain complexes $C^{*,*}$ are defined similarly, with differentials $\delta^h \colon C^{i,j} \to C^{i+1,j}$ and $\delta^v \colon C^{i,j} \to C^{i,j+1}$.

Definition 2.2.18. Let $\mathcal{C}$ be an additive category which admits products and coproducts. Let $C_{*,*}$ be a double complex in $\mathcal{C}$.

- The (product) total complex $\mathrm{Tot}^{\prod}(C_{*,*})$ is defined as the chain complex with entries
$$\mathrm{Tot}^{\prod}(C_{*,*})_n = \prod_{i+j=n} C_{i,j}$$
and differential $d = d^v + d^h$, meaning that for $(x_{i,j})_{i+j=n} \in \prod_{i+j=n} C_{i,j}$
$$d((x_{i,j})_{i+j=n}) = (d^h(x_{i+1,j}) + d^v(x_{i,j+1}))_{i+j=n-1}\,.$$

- The (coproduct) total complex $\mathrm{Tot}^{\oplus}(C_{*,*})_*$ is defined as the chain complex with entries
$$\mathrm{Tot}^{\oplus}(C_{*,*})_n = \bigoplus_{i+j=n} C_{i,j}$$
and differential $d = d^v + d^h$.

Similarly, if $C^{*,*}$ is a double cochain complex, then $\mathrm{Tot}^{\prod}(C^{*,*})$ is the cochain complex with $\mathrm{Tot}^{\prod}(C^{*,*})^n = \prod_{i+j=n} C^{i,j}$ and $\delta = \delta^v + \delta^h$.

We note that in the total complex $d^2 = (d^h + d^v)^2 = (d^h)^2 + d^h d^v + d^v d^h + (d^v)^2 = 0$, so that it indeed defines a chain complex. We note that if $C_{*,*}$ is a bounded double chain complex, then $\mathrm{Tot}^{\prod}(C_{*,*}) \simeq \mathrm{Tot}^{\oplus}(C_{*,*})$, and we write $\mathrm{Tot}(C_{*,*}) \coloneqq \mathrm{Tot}^{\prod}(C_{*,*})$.

Lemma 2.2.19. *Let $\mathcal{C}$ be an abelian category with products and coproducts. Let $C_{*,*}$ be a bounded double chain complex in $\mathcal{C}$ such that $(C_{*,j}, d^h)$ is an exact chain complex for all $j \in \mathbb{Z}$. Then $\mathrm{Tot}(C_{*,*})$ is an exact chain complexes.*

Proof. By shifting indices, it suffices to show that $\mathrm{Tot}(C_{*,*})$ is exact in position 0. Let $i \in \mathbb{Z}$ be minimal with $C_{i,-i} \neq 0$. Let $X = (x_{j,-j})_{j\in\mathbb{Z}} \in \bigoplus_{j\in\mathbb{Z}} C_{j,-j}$ and suppose that $X \in \ker(d)$, meaning that
$$d^h(x_{j,-j}) + d^v(x_{j-1,-(j-1)}) = 0 \qquad \forall j \in \mathbb{Z}\,.$$
We must show that $X \in \mathrm{Im}(d)$.

Since $x_{i-1,-(i-1)} = 0$, we see that $d^h(x_{i,-i}) = 0$. Since $(C_{*,-i}, d^h)$ is exact, there thus exists $y_i \in C_{i+1,-i}$ with $d^h(y_i) = x_{i,-i}$. Then
$$d^h(x_{i+1,-(i+1)} - d^v(y_i)) = d^h(x_{i+1,-(i+1)}) - d^v d^h(y_i) = d^h(x_{i+1,-(i+1)}) - d^v(x_{i,-i}) = 0\,.$$
Iteratively, we can thus find $y_j \in C_{j+1,-j}$ with $d^h(y_j) = x_{j,-j} - d^v(y_{j-1})$, or equivalently $d^h(y_j) + d^v(y_{j-1}) = x_{j,-j}$. Then $Y = (y_j)_j \in \bigoplus_{j\in\mathbb{Z}} C_{j+1,-j}$ satisfies $d(Y) = X$. $\square$

Theorem 2.2.20 (Balancing Ext and Tor)**.**

(i) Let $\mathcal{C}$ be an abelian category with enough projectives and enough injectives. Then for all $A, B \in \mathcal{C}$ and $n \geq 0$ there exists an isomorphism
$$\mathrm{Ext}^n_{\mathcal{C}}(A, B) = R^n \operatorname{Hom}_{\mathcal{C}}(\text{-}, B)(A) \simeq R^n \operatorname{Hom}_{\mathcal{C}}(A, \text{-})(B)\,.$$

(ii) Let R be a ring, $M \in \mathrm{RMod}_R$ and $N \in \mathrm{LMod}_R$. Then
$$\mathrm{Tor}^n_R(M, N) = L_n(M \otimes_R \text{-})(N) \simeq L_n(\text{-} \otimes_R N)(M)\,.$$

Proof. We will only prove part (i), part (ii) follows from a similar argument and is left as an exercise. Let $A, B \in \mathcal{C}$ and choose an injective resolution $B \to I^*$ and a projective resolution $P_* \to A$. Denote the codifferential of I^* by δ and the differential of P_* by d. Then there is a double cochain complex $\operatorname{Hom}_{\mathcal{C}}(P_*, I^*)^{*,*}$ with

$$\operatorname{Hom}_{\mathcal{C}}(P_*, I^*)^{i,j} = \operatorname{Hom}_{\mathcal{C}}(P_i, I^j)$$

$$\delta^h_{i,j} = (\text{-}) \circ d\colon \operatorname{Hom}_{\mathcal{C}}(P_i, I^j) \to \operatorname{Hom}_{\mathcal{C}}(P_{i-1}, I^j)$$

and

$$\delta^v_{i,j} = (-1)^{i+j+1}\delta \circ (\text{-})\colon \operatorname{Hom}_{\mathcal{C}}(P_i, I^j) \to \operatorname{Hom}_{\mathcal{C}}(P_i, I^{j-1})\,.$$

Similarly, denoting by

$$\tilde{I}^* = (\cdots \to 0 \to B \to I^0 \to I^1 \to \dots)$$

and

$$\tilde{P}_* = (\cdots \to P_1 \to P_0 \to A \to 0 \to \dots)$$

the augmented (co)chain complexes (which are exact) (shifted to lie in degrees ≥ -1), we obtain two bounded double cochain complexes

$$\operatorname{Hom}_{\mathcal{C}}(P_*, \tilde{I}^*)^{*,*}$$

and

$$\operatorname{Hom}_{\mathcal{C}}(\tilde{P}_*, I^*)^{*,*}$$

whose totalizations are exact by Lemma 2.2.19. There are short exact sequences of cochain complexes

$$\operatorname{Tot}\operatorname{Hom}_{\mathcal{C}}(P_*, I^*)^{*,*} \longrightarrow \operatorname{Tot}\operatorname{Hom}_{\mathcal{C}}(P_*, \tilde{I}^*)^{*,*} \longrightarrow \operatorname{Hom}_{\mathcal{C}}(P_*, B)[1] \tag{27}$$

and

$$\operatorname{Tot}\operatorname{Hom}_{\mathcal{C}}(P_*, I^*)^{*,*} \longrightarrow \operatorname{Tot}\operatorname{Hom}_{\mathcal{C}}(\tilde{P}_*, I^*)^{*,*} \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, I^*)[1] \tag{28}$$

where the convention for the shift of cochain complexes is $(X^*[1])^n = X^{n+1}$. For $n \geq 0$, note that

$$\bigoplus_{i+j=n} \operatorname{Hom}_{\mathcal{C}}(P_i, \tilde{I}^j) = \operatorname{Hom}_{\mathcal{C}}(P_{n+1}, \tilde{I}^{-1}) \oplus \bigoplus_{i+j=n} \operatorname{Hom}_{\mathcal{C}}(P_i, I^j)$$

where $\tilde{I}^{-1} = B$, and the sequence (27) is in degree n isomorphic to the split SES

$$\bigoplus_{i+j=n} \operatorname{Hom}_{\mathcal{C}}(P_i, I^j) \to \operatorname{Hom}_{\mathcal{C}}(P_{n+1}, B) \oplus \bigoplus_{i+j=n} \operatorname{Hom}_{\mathcal{C}}(P_i, I^j) \to \operatorname{Hom}_{\mathcal{C}}(P_{n+1}, B)\,.$$

The LES of (27) reads as

$$\cdots \to \underbrace{H^{n-1}\operatorname{Tot}\operatorname{Hom}_{\mathcal{C}}(P_*, \tilde{I}^*)^{*,*}}_{\simeq 0} \to \underbrace{H^{n-1}(\operatorname{Hom}_{\mathcal{C}}(P_*, B)[1])}_{\simeq H^n \operatorname{Hom}_{\mathcal{C}}(P_*,B)} \to H^n \operatorname{Tot}\operatorname{Hom}_{\mathcal{C}}(P_*, I^*)^{*,*} \to \underbrace{\dots}_{\simeq 0}$$

so that
$$H^n \operatorname{Hom}_{\mathcal{C}}(P_*, B) \simeq H^n \operatorname{Tot} \operatorname{Hom}_{\mathcal{C}}(P_*, I^*)^{*,*}$$
Similarly, the LES of (28) shows that
$$H^n \operatorname{Hom}_{\mathcal{C}}(A, I^*) \simeq H^n \operatorname{Tot} \operatorname{Hom}_{\mathcal{C}}(P_*, I^*)^{*,*} .$$
Combining these two isomorphisms yields the desired isomorphism. □

Remark 2.2.21. Passing to the totalization defines an exact functor from the category of double complexes to chain complexes. The SES of chain complexes (27) arises from a SES of double complexes.

Example 2.2.22. An injective resolution of $\mathbb{Z} \in \mathrm{Ab}$ is given by $I^* = \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \to \dots$. We find that
$$\operatorname{Ext}^n_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \simeq H^n \operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) .$$
For any abelian group A, we have that
$$\operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, A) \simeq \{x \in A | 2x = 0\} \subset A$$
describes the subgroups of elements of order 2.

We have $\operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) \simeq 0$ since the only element $x \in \mathbb{Q}$ satisfying $2x = 0$ is $x = 0$. Further, we have
$$\operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \{(0, 1 \mapsto 0), (0 \mapsto 0, 1 \mapsto \frac{1}{2})\} ,$$
since $0, \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$ are the only element of order 2. Thus
$$\operatorname{Ext}^*_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \simeq \begin{cases} 0 & n \neq 1 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 , \end{cases}$$
as expected from a computation using the projective resolution $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ of $\mathbb{Z}/2\mathbb{Z}$.

2.3 Group cohomology

2.3.1 Invariants and coinvariants

Let G be a group. A G-action on an abelian group A consists of a morphism of sets $(\text{-}) \cdot (\text{-}) \colon G \times A \to A$, such that

- $g \cdot (g' \cdot (\text{-})) = (gg') \cdot (\text{-})$ for all $g, g' \in G$,
- $1_G \cdot (\text{-}) = \mathrm{id}_A$, and
- $g \cdot (\text{-}) \colon A \to A$ is a morphism of abelian groups for all $g \in G$.

In this case, we also call A a G-module. A morphism of G-modules consists of a morphism $f\colon A \to B$ of abelian groups which commutes with the group action, meaning that for all $g \in G$ the diagram

$$\begin{tikzcd} A \arrow[r, "g\cdot(\text{-})"] \arrow[d, "f"] & A \arrow[d, "f"] \\ B \arrow[r, "g\cdot(\text{-})"] & B \end{tikzcd}$$

commutes.

We denote by G- Mod the category of G-modules.

Recall that BG denotes the category with a single object $*$ and $\mathrm{Hom}_{BG}(*,*) = G$. A G-module A can equivalently be described as a functor $\varrho_A\colon BG \to \mathrm{Ab}$, mapping $*$ to A and $g\colon * \to *$ to $g \cdot (\text{-})\colon A \to A$. .

Lemma 2.3.1. *There exists an isomorphism of categories*

$$\varrho_{(\text{-})}\colon G\text{-}\mathrm{Mod} \simeq \mathrm{Fun}(BG, \mathrm{Ab})\,.$$

Proof. The bijection on objects is clear. A morphism of G-modules $f\colon A \to B$ is determined by the underlying morphism of abelian groups, and similarly a natural transformation $\eta\colon \varrho_A \to \varrho_B$ between functors $\varrho_A, \varrho_B\colon BG \to \mathrm{Ab}$ is determined by the morphism of abelian groups $\eta_*\colon \varrho_A(*) = A \to \varrho_B(*) = B$. This induces a bijection between G-module morphisms and corresponding natural transformations. ☐

Definition 2.3.2. Let G be a group and A a G-module.

(1) The invariant subgroup $A^G \subset A$ is the subgroup with elements

$$A^G = \{a \in A | g \cdot a = a \text{ for all } g \in G\}\,.$$

(2) The coinvariants A_G of A is the quotient abelian group

$$A_G = A/\langle\{(g\cdot a - a)\}_{a\in A, g\in G}\rangle\,,$$

where $\langle\{(g\cdot a - a)\}_{a\in A, g\in G}\rangle \subset A$ denotes the smallest subgroup of A containing the elements $\{(g\cdot a - a)\}_{a\in A, g\in G} \subset A$.

Lemma 2.3.3. *Let G be a group.*

(1) For each G-module A with corresponding functor $\varrho_A\colon BG \to \mathrm{Ab}$, there exists an isomorphism of abelian groups

$$A^G \simeq \lim \varrho_A\,.$$

The passage to the invariant subgroup defines a functor $(\text{-})^G\colon \mathrm{G\text{-}Mod} \to \mathrm{Ab}$ and the above isomorphism extends to a natural equivalence $\lim_{BG} \circ \varrho_{(\text{-})} \simeq (\text{-})^G$:

$$\begin{tikzcd} G\text{-}\mathrm{Mod} \arrow[rr, "\simeq", "\varrho_{(\text{-})}"'] \arrow[dr, "(\text{-})^G"'] & & \mathrm{Fun}(BG, \mathrm{Ab}) \arrow[dl, "\lim_{BG}"] \\ & \mathrm{Ab} & \end{tikzcd}$$

(2) *For each G-module A with corresponding functor $\varrho_A\colon BG \to \mathrm{Ab}$, there exists an isomorphism of abelian groups*

$$A_G \simeq \operatorname{colim} \varrho_A \,.$$

The passage to the coinvariants defines a functor $(\text{-})_G\colon \mathrm{G\text{-}Mod} \to \mathrm{Ab}$ and the above isomorphism extends to a natural equivalence $(\text{-})_G \simeq \operatorname{colim}_{BG} \circ \varrho_{(\text{-})}$.

Proof. Exercise. □

We next illustrate a fact from category theory, which is a consequence of the interchange of limits from Corollary 1.3.24. This fact will allow us to deduce that the functor $(\text{-})^G$ is right exact from general principles. Of course, one could also proof this in a direct way, but the more conceptual proof has multiple advantages. For instance, it easier to remember and can more easily be generalized.

Lemma 2.3.4. *Let $K, L, \mathcal{C}$ be a categories.*

(1) *Let $F\colon K \to \mathrm{Fun}(L, \mathcal{C})$ be a diagram with a cone $(X, \{p_k\}_{k\in K})$. Then $(X, \{p_k\})_{k\in K}$ is a limit cone if and only if $(X(l), \{p_k(l)\}_{k\in K})$ is a limit cone in $\mathcal{C}$ for all $l \in L$.*

(2) *Suppose that $\mathcal{C}$ admits L-indexed limits. Then the limit functor $\lim_L\colon \mathrm{Fun}(L, \mathcal{C}) \to \mathcal{C}$ preserves limits.*

Proof sketch. The proof of part (1) is left as an exercise. For part (2), let $F\colon K \to \mathrm{Fun}(L, \mathcal{C})$ be a diagram. The functor $F\colon K \to \mathrm{Fun}(L, \mathcal{C})$ determines a functor $\tilde{F}(\text{-}, \text{-})\colon K \times L \to \mathcal{C}$, given on objects $(k, l) \in K \times L$ by $\tilde{F}(k, l) = F(k)(l)$.

By part (1) and the assumption that $\mathcal{C}$ admits L-indexed limit, it admits a limit cone $(\lim_K(F), \{p_k\}_{k\in K})$. We must show that

$$(\lim_L(\lim_K(F)), \{\lim_L(p_k)\}_{k\in K}) \tag{29}$$

is a limit cone over $\lim_L \circ F$. Part (1) shows that $\lim_L \circ F \simeq \lim_{l\in L} \tilde{F}(\text{-}, l)\colon K \to \mathcal{C}$.

The proof of Proposition 1.3.23 shows that cones over $\lim_{k\in K} \tilde{F}(k, \text{-})$ are in bijection with cones over $\tilde{F}(\text{-}, \text{-})$, which in turn are in bijection with cones over $\lim_{l\in L} \tilde{F}(\text{-}, l)$, and furthermore these bijection restrict to bijections between the collections of limit cones. Spelling out the cone over $\lim_{k\in K} \tilde{F}(k, \text{-})\colon L \to \mathcal{C}$ corresponding to the cone (29), one finds it to be a limit cone, which thus implies that the cone (29) is a limit cone. □

Corollary 2.3.5. *The functor $(\text{-})^G\colon G\text{-}\mathrm{Mod} \to \mathrm{Ab}$ preserves limits, and is in particular left exact, and the functor $(\text{-})_G\colon G\text{-}\mathrm{Mod} \to \mathrm{Ab}$ preserves colimits, and is in particular right exact.*

Proof. It follows from Corollary 1.3.24 that limit functors preserve limits and colimit functors preserve colimits. The assertion thus follows from Lemma 2.3.3. □

2.3.2 Group rings

Definition 2.3.6. Let G be a group and R a ring. The group ring $R[G]$ is defined as the ring with

- underlying set $\bigoplus_{g\in G} R$, so that we write elements as sums $\sum_{g\in G}(g, r_g)$ with finitely many non-zero summands,
- multiplication defined by

$$\left(\sum_{g_1\in G}(g_1, r_{g_1})\right)\cdot\left(\sum_{g_2\in G}(g_2, r'_{g_2})\right) = \sum_{g\in G}\ \sum_{(g_1,g_2)\in G\times G, g_1\cdot g_2=g}(g, r_{g_1}\cdot r'_{g_2})$$

- and unit given by $1_{RG} = (1_G, 1_R)$.

In the following, we will be interested in the group ring in the case $R = \mathbb{Z}$.

Example 2.3.7. The group ring $\mathbb{Z}[\mathbb{Z}]$ of the group of integers $\mathbb{Z}$ is isomorphic to the ring of Laurent polynomials $\mathbb{Z}[x^{\pm}]$:

$\mathbb{Z}[x^{\pm}]$ is free as an abelian group, with generating set consisting of the symbols $\{x^k\}_{k\in\mathbb{Z}}$. Thus, elements of $\mathbb{Z}[x^{\pm}]$ are Laurent polynomials, meaning expressions of the form $p = \sum_{k\in\mathbb{Z}} n_k x^k$ with finitely many non-zero terms in the sum. The multiplication is the usual one of polynomials, meaning that

$$\left(\sum_{k\in\mathbb{Z}} n_k x^k\right)\cdot\left(\sum_{k\in\mathbb{Z}} m_k x^k\right) = \sum_{k\in\mathbb{Z}}\sum_{k=i+j} n_i m_j x^k\,.$$

The isomorphism is given by the assignment

$$\mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[x^{\pm}], \quad \sum_{a\in\mathbb{Z}}(a, n_a) \mapsto \sum_{a\in\mathbb{Z}} n_a x^a\,.$$

The polynomial ring $\mathbb{Z}[x]$ is the subring of $\mathbb{Z}[x^{\pm}]$ generated by the elements $\{x^k\}_{k\in\mathbb{N}}$. Since $\mathbb{N}$ is not a group (only a monoid), $\mathbb{Z}[x]$ is not itself a group ring (one can still define $\mathbb{Z}[\mathbb{N}]$ and call it a monoid ring).

Remark 2.3.8. For any group G there exists a ring homomorphism $\psi\colon \mathbb{Z}[G] \to \mathbb{Z}, \sum_g(g, n_g) \mapsto \sum_g n_g$. Using ψ, we can consider $\mathbb{Z}$ as a $\mathbb{Z}[G]$-$\mathbb{Z}$-bimodule or a $\mathbb{Z}$-$\mathbb{Z}[G]$-bimodule (with the action of $a \in \mathbb{Z}[G]$ given by multiplying with $\psi(a) \in \mathbb{Z}$).

Exercise 2.3.9. Show that $\mathbb{Z}[G] \simeq \mathbb{Z}[G]^{\mathrm{op}}$. Note that $\mathbb{Z}[G]$ is in general not a commutative ring.

Lemma 2.3.10. *There exists an isomorphism of categories*

$$G\text{-}\mathrm{Mod} \simeq \mathrm{LMod}_{\mathbb{Z}[G]} \simeq \mathrm{RMod}_{\mathbb{Z}[G]}\,.$$

Proof sketch. Given a G-module A, we obtain an action $(\text{-}) \cdot (\text{-}) \colon \mathbb{Z}[G] \times A \to A$ by

$$\left(\sum_{g \in G} (g, n_g)\right) \cdot a = \sum_{g \in G} n_g \cdot (g \cdot a)\,.$$

Conversely, given an action $\mathbb{Z}[G] \times A \to A$, restricting along the inclusion of sets $G \subset \mathbb{Z}[G], g \mapsto (g, 1)$, induces a G-action on A. These assignments readily extend to inverse functors. The second isomorphisms follows from $\mathbb{Z}[G] \simeq \mathbb{Z}[G]^{\mathrm{op}}$. □

We will be abusive in notation, identifying G-modules and $\mathbb{Z}[G]$-modules.

Remark 2.3.11. The $\mathbb{Z}[G]$-module $\mathbb{Z}$ from Remark 2.3.8 corresponds to the G-module with the trivial action $g \cdot n = n$ for all $g \in G$. This G-module is also called the trivial G-module.

Lemma 2.3.12. *Let $A \in G\text{-}\mathrm{Mod}$. Then*

$$A^G \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

and

$$A_G \simeq A \otimes_{\mathbb{Z}[G]} \mathbb{Z}$$

natural in A.

Proof. We obtain a morphism of abelian groups $A^G \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ by the assignment $a \mapsto f_a = (n \mapsto n \cdot a)$. The inverse of this morphism is $(f \colon \mathbb{Z} \to A) \mapsto f(1)$. One readily checks that these are inverse isomorphism of abelian groups.

For the second isomorphism, consider the epimorphism of $\mathbb{Z}[G]$-modules

$$\psi \colon \mathbb{Z}[G] \to \mathbb{Z}, \quad \sum_g (g, n_g) \mapsto \sum_g n_g\,.$$

The kernel of ψ is given by the $\mathbb{Z}[G]$-submodule of $\mathbb{Z}[G]$ generated by expressions of the form $\{(1_G, 1) - (g, 1)\}_{g \in G}$. Using that $A \otimes_{\mathbb{Z}[G]} (\text{-})$ is right exact, we obtain an exact sequence

$$A \otimes_{\mathbb{Z}[G]} \ker(\psi) \to \underbrace{A \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]}_{\simeq A} \to A \otimes_{\mathbb{Z}[G]} \mathbb{Z} \to 0\,.$$

Since $A \otimes_{\mathbb{Z}[G]} \ker(\psi)$ is generated by pure tensors, it is generated by elements of the form

$$a \otimes_{\mathbb{Z}[G]} ((1_G, 1) - (g, 1)) = (a - g.a) \otimes_{\mathbb{Z}[G]} (1_G, 1)$$

with $a \in A$ and $g \in G$. We thus find that the cokernel $A \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ above is isomorphic to A_G.

The naturality of these isomorphisms is left as an exercise. □

We can thus understand G-modules and invariants in multiple ways:

- explicitly, as set-theoretic constructions,
- as a special case of functor categories and limit functors,
- and from the perspective of modules over rings (group rings) and tensor products of bimodules.

Proposition 2.3.13. *The category* G-Mod *has enough injectives and projectives.*

Proof. This follows from Lemma 2.3.10 using the fact that module categories have enough injectives and projectives. ☐

2.3.3 Group cohomology and the bar construction

Definition 2.3.14. Let G be a group.

(1) For $A \in G$-Mod and $n \geq 0$, we define

$$H^n(G, A) \coloneqq R^n\Big((\text{-})^G\Big)(A)$$

and

$$H_n(G, A) \coloneqq L_n\Big((\text{-})_G\Big)(A)\,.$$

(2) The n-th group cohomology of G is defined as

$$H^n(G) \coloneqq H^n(G, \mathbb{Z})$$

where $\mathbb{Z}$ is the trivial G-module.

(3) The n-th group homology of G is similarly defined as

$$H_n(G) \coloneqq H_n(G, \mathbb{Z})\,.$$

Remark 2.3.15. The natural equivalences from Lemma 2.3.12 imply that the corresponding derived functors are also equivalent:

$$H^n(G, A) \simeq \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

and

$$H_n(G, A) \simeq \mathrm{Tor}_n^{\mathbb{Z}[G]}(A, \mathbb{Z})\,.$$

To compute group cohomology and homology, we can employ a specific projective resolution of $\mathbb{Z} \in G$-Mod called the bar construction.

Definition 2.3.16. Let R be a ring, $M \in \mathrm{RMod}_R$ and $N \in \mathrm{LMod}_R$. Denote the linear maps induced by the bilinear action maps by $\mu_M \colon M \otimes R \to M$ and $\mu_N \colon R \otimes N \to N$ and the linear map induced by the multiplication in R by $\mu_R \colon R \otimes R \to R$. The two-sided bar construction is the chain complex $B_*(M, R, N)$ with

$$B_n(M, R, N) = M \otimes R^{\otimes n} \otimes N$$

and differential $d_n = \sum_{i=0}^n (-1)^i d_n^i$, where

$$d_n^i = \begin{cases} \mu_M \otimes \mathrm{id}_R \otimes \cdots \otimes \mathrm{id}_R \otimes \mathrm{id}_N & i = 0 \\ \mathrm{id}_M \otimes \mathrm{id}_R \otimes \cdots \otimes \underbrace{\mu_R}_{\text{pos. } i} \otimes \cdots \otimes \mathrm{id}_R \otimes \mathrm{id}_N & i \in [1, n-1] \\ \mathrm{id}_M \otimes \mathrm{id}_R \otimes \cdots \otimes \mathrm{id}_R \otimes \mu_n & i = n\,. \end{cases}$$

Remark 2.3.17.

(1) We can write a pure tensor in degree n in $B_n(M, R, N)$ as $x \otimes r_1 \otimes \cdots \otimes r_n \otimes y$ with $x \in M, y \in N$ and $r_i \in R$. On such a pure tensor, the differential acts as

$$\begin{aligned} d_n(x \otimes r_1 \otimes \cdots \otimes r_n \otimes y) = (x \cdot r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y) \\ + \sum_{i=1}^{n-1} (-1)^i x \otimes \dots r_{i-1} \otimes r_i \cdot r_{i+1} \otimes r_{i+2} \otimes \dots y \\ + (-1)^n (x \otimes r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n \cdot y)\,. \end{aligned}$$

(2) We have $d_{n-1}^i d_n^j = d_{n-1}^{j-1} d_n^i$ if $i < j$, as one readily checks on pure tensors. This implies that

$$\begin{aligned} d^2 = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_{n-1}^i d_n^j &= \sum_{0 \le i < j \le n} (-1)^{i+j} d_{n-1}^i d_n^j + \sum_{0 \le j \le i \le n} (-1)^{i+j} d_{n-1}^i d_n^j \\ &= \sum_{0 \le i < j \le n} (-1)^{i+j} d_{n-1}^{j-1} d_n^i + \sum_{0 \le j \le i \le n-1} (-1)^{i+j} d_{n-1}^i d_n^j \\ &= \sum_{0 \le i \le j' \le n-1} (-1)^{i+j'+1} d_{n-1}^{j'} d_n^i + \sum_{0 \le j \le i \le n-1} (-1)^{i+j} d_{n-1}^i d_n^j \\ &= 0 \end{aligned}$$

where in the second to last step, we replaced the index j in the first sum with the index $j' = j - 1$. This shows that $B_*(M, R, N)$ indeed defines a chain complex.

We note that $B_*(R, R, N)$ canonically defines a chain complex in LMod_R, where the left R-action is given on the left-most components. On pure tensors, the action is thus given as

$$a \cdot (r_0 \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y) = (a \cdot r_0) \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y\,,$$

where $a, r_i \in R$ and $y \in N$.

Lemma 2.3.18. *Let R be a ring and $N \in \mathrm{LMod}_R$.*

(1) With the augmentation map $\epsilon = \mu_N \colon B_0(R, R, N) = R \otimes N \to N$, the chain complex $B_*(R, R, N)$ is a resolution of N.

(2) If R, M are free as abelian groups, then $B_*(R, R, N) \to N$ is a free resolution in LMod_R.

Proof. To show part (1), we show that the identity of the augmented chain complex $\tilde{B}_*(R, R, N)$ with $d_0 = \epsilon$ is null homotopic. This implies that $\tilde{B}_*(R, R, N)$ is exact, by Lemma 1.5.19 and using that $H_n(\mathrm{id}_{\tilde{B}_*(R,R,N)}) = \mathrm{id}_{H_n(\tilde{B}_*(R,R,N))}$. Let $\phi\colon \mathbb{Z} \to R$ be the unique ring homomorphism. For $n \geq -1$, we define $H_n\colon \tilde{B}_n(R, R, N) \to \tilde{B}_{n+1}(R, R, N)$ as

$$H_n = \phi \otimes (\mathrm{id}_R)^{\otimes n+1} \otimes \mathrm{id}_N$$

Then $dH + Hd = \mathrm{id}$ as one checks on pure tensors: Given $X = (r_0 \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y) \in \tilde{B}_*(R, R, N)_n$, we have

$$dH(X) = X + \sum_{i=1}^{n} (-1)^i d^i_{n+1}(1_R \otimes r_0 \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y)$$

and

$$Hd(X) = \sum_{i=0}^{n-1} (-1)^i 1_R \otimes d^i_n(r_0 \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y)$$

The desired equality follows from

$$d^{i+1}_{n+1}(1_R \otimes r_0 \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y) = 1_R \otimes d^i_n(r_0 \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y)\,.$$

For part (2), we note that if R, M are free, then $R^{\otimes n} \otimes N \simeq \mathbb{Z}^{\oplus I}$ for some set I, as follows from Proposition 1.6.40 (applied to the $\mathbb{Z}$-linear tensor product). We thus find that

$$B_n(R, R, M) = R \otimes R^{\otimes n} \otimes N \simeq R \otimes \mathbb{Z}^{\oplus I} \simeq R^{\oplus I}$$

is a free R-module. □

Corollary 2.3.19. *Let $A \in G\text{-}\mathrm{Mod} \simeq \mathrm{LMod}_{\mathbb{Z}[G]} \simeq \mathrm{RMod}_{\mathbb{Z}[G]}$. Then*

$$H^n(G, A) \simeq H^n \operatorname{Hom}_{\mathbb{Z}[G]}(B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}), A)$$

and

$$H_n(G, A) \simeq H_n(A \otimes_{\mathbb{Z}[G]} B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}))\,.$$

Proof. Combine Remark 2.3.15 and Lemma 2.3.18. □

Proposition 2.3.20. *Let G be a group. Then $H_1(G) \simeq G/[G, G]$, where $[G, G] \subset G$ is the subgroup generated by elements of the form $\{gh(hg)^{-1}\}_{g,h \in G}$ (called the commutator subgroup).*

The abelian group $G/[G, G]$ is also called the *abelianization* of G.

Proof of Proposition 2.3.20. Since $B_n(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) = \mathbb{Z}[G]^{\otimes n+1} \otimes \mathbb{Z} \simeq \mathbb{Z}[G]^{\otimes n+1}$, the chain complex $\mathbb{Z} \otimes_{\mathbb{Z}[G]} B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$ is of the following form

$$\cdots \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \otimes \mathbb{Z}[G] \otimes \mathbb{Z}[G] \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \otimes \mathbb{Z}[G] \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]$$

which is in turn isomorphic to

$$\cdots \to \mathbb{Z}[G] \otimes \mathbb{Z}[G] \to \mathbb{Z}[G] \to \mathbb{Z}$$

The differential $\mathbb{Z}[G] \to \mathbb{Z}$ vanishes. The differential $d\colon \mathbb{Z}[G] \otimes \mathbb{Z}[G] \to \mathbb{Z}[G]$ acts on a pure tensor $(g,1) \otimes (h,1)$ as

$$d((g,1) \otimes (h,1)) = (h,1) - (gh,1) + (g,1) = (g,1) + (h,1) - (gh,1)\,.$$

The first homology is thus given by the quotient abelian group

$$\mathbb{Z}[G]/\langle (g,1) + (h,1) - (gh,1) \rangle$$

We note that in $\mathbb{Z}[G]/\langle (g,1) + (h,1) - (gh,1) \rangle$ there are equalities

$$[\sum_{g\in G} (g, n_g)] = [\sum_{g\in G} n_g \cdot (g,1)] = [\sum_{g\in G} (\prod_{n_g} g, 1)] = [(\prod_{g\in G} \prod_{n_g} g, 1)]\,,$$

where the first equality follows from the definition of addition in $\mathbb{Z}[G]$ and the second and third equalities are relations in the quotient. We thus have two inverse isomorphisms of abelian groups:

$$G/[G,G] \to \mathbb{Z}[G]/\langle (g,1) + (h,1) - (gh,1) \rangle, \quad [g] \mapsto (g,1)$$
$$\mathbb{Z}[G]/\langle (g,1) + (h,1) - (gh,1) \rangle \to G/[G,G], \quad [\sum_{g\in G} (g,n_g)] \mapsto [\prod_{g\in G} \prod_{n_g} g]\,.$$

□

The goal is to describe $H^2(G,A)$ in the next section. For this, we next describe the 2-cocycles $Z^2(\mathrm{Hom}_{\mathbb{Z}[G]}(B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}), A))$.

Lemma 2.3.21. *Let G be a group and $A \in G\text{-}\mathrm{Mod}$. Then there is an isomorphism of abelian groups*

$$\begin{aligned} &Z^2(\,\mathrm{Hom}_{\mathbb{Z}[G]}(B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}), A)) \\ &\qquad \simeq \{[\text{-},\text{-}]\colon G \times G \to A \mid 0 = f[g,h] - [fg,h] + [f,gh] - [f,g]\ \forall f,g,h \in G\} \end{aligned}$$

Proof. We have $B_2(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \simeq \mathbb{Z}[G]^{\otimes 3}$, which is free as a left $\mathbb{Z}[G]$-module, and thus

$$\mathrm{Hom}_{\mathbb{Z}[G]}(B_2(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}), A) \simeq \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[G]^{\otimes 2}, A) \simeq \mathrm{Hom}_{\mathrm{Set}}(G \times G, A)\,, \quad (30)$$

where the later isomorphism uses that $\mathbb{Z}[G]$ is a free abelian group with basis G. Given $[\text{-},\text{-}]\colon G \times G \to A$, we must compute the codifferential of the corresponding $\mathbb{Z}[G]$-module homomorphism $B_2(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \to A$. We have

$$d_3 = d_3^1 - d_3^2 + d_3^3 - d_3^4\colon B_3(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \to B_2(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})\,,$$

There are canonical inclusions $G^{\times 4} \subset \mathbb{Z}[G]^{\otimes 4} \simeq B_3(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$, and $G^{\times 3} \subset B_2(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$, and the maps d_3^i restrict to maps $G^{\times 4} \to G^{\times 3}$, acting on $(1_G, f, g, h) \in G^{\times 4}$ as follows:

$$\begin{aligned} d_3^1(1_G, f, g, h) &= (f, g, h) \\ d_3^2(1_G, f, g, h) &= (1_g, fg, h) \\ d_3^3(1_G, f, g, h) &= (1_g, f, gh) \\ d_3^4(1_G, f, g, h) &= (1_g, f, g) \end{aligned}$$

Under the isomorphism from Equation (30), a morphism of sets $[\text{-}, \text{-}]\colon G \times G \to A$ amounts to a $Z[G]$-module homomorphism $\alpha\colon Z[G]^{\otimes 3} \to A$ acting on $(f, g, h) \in G^{\times 3} \subset Z[G]^{\otimes 3}$ as $\alpha(f, g, h) = f[g, h]$, where the multiplication with f from the left uses the G-action on $[g, h] \in A$. Combining this observation with the above description of the differentials d_3^i, and the fact that $\{1_G\} \times G^{\times 3} \subset \mathbb{Z}[G]^{\otimes 4}$ (freely) generates as a $\mathbb{Z}[G]$-module, the assertion follows. □

2.3.4 Group extensions and $H^2(G, A)$

Definition 2.3.22. A short exact sequence of groups

$$1 \to G_1 \xrightarrow{\iota} G_2 \xrightarrow{\pi} G_3 \to 1$$

consists of two group homomorphisms ι, π, such that ι is injective, π is surjective, and $\mathrm{Im}(\iota) = \ker(\pi)$.

Note that the category Grp is not abelian. In a short exact sequence of groups, $G_1 \simeq \mathrm{Im}(\iota) = \ker(\pi) \subset G_2$ is a normal subgroup as a kernel of a group homomorphism.

In a group extension, 1 denotes the group with a single element, written multiplicatively. If G_1 is abelian, we write the sequence as

$$0 \to G_1 \xrightarrow{\iota} G_2 \xrightarrow{\pi} G_3 \to 1\,,$$

using instead the additive notation for the trivial abelian group. In the following, we will only be interested in the case that $G_1 = A$ is abelian.

Construction 2.3.23. Consider an extension of groups

$$0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

and let $g \in G$. Choose any $e \in E$ with $\pi(e) = g$. Since $A \subset E$ is normal, conjugation by e defines a morphism $e(\text{-})e^{-1}\colon A \to A$. If we choose a different $e' \in E$ with $\pi(e') = g$, then $\pi(e^{-1}e') = 1_G$, and so there exists $a \in A$ with $e \cdot a = e'$. Thus $e'(\text{-})(e')^{-1} = ea(\text{-})a^{-1}e^{-1} = e(\text{-})e^{-1}\colon A \to A$, since A is commutative.

We hence obtain a well-defined morphism $G \times A \to A$, mapping a pair (g, a) to $eae^{-1} \in \mathrm{Im}(\iota) \simeq A$, where $e \in E$ is any element with $\pi(e) = g$.

One readily checks that the above defined morphism $G \times A \to A$ defines a group action:

Lemma 2.3.24. *Let*

$$0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

be an extension of groups. Then the morphism $G \times A \to A$ from Construction 2.3.23 defines an action of G on A.

Definition 2.3.25. Let G be a group and $A \in G\text{-}\mathrm{Mod}$. An extension of G by A consists of a short exact sequence of groups

$$0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

such that the arising G-action on A from Lemma 2.3.24 coincides with the given G-action on A.

Definition 2.3.26. Let G be a group and $A \in G\text{-}\mathrm{Mod}$. Two extensions of G by A

$$0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

$$0 \to A \xrightarrow{\iota'} E' \xrightarrow{\pi'} G \to 1$$

are said to be equivalent if there exists a group homomorphism $\alpha\colon E \to E'$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\mathrm{id}_A"'] & E \arrow[r, "\pi"] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "\mathrm{id}_G"] & 1 \\
0 \arrow[r] & A \arrow[r, "\iota'"] & E' \arrow[r, "\pi'"] & G \arrow[r] & 1
\end{tikzcd}$$

We remark that in this case α must be an isomorphism.

The main result on group cohomology will be the following description of the set of equivalence classes of extensions of a group G by $A \in G\text{-}\mathrm{Mod}$, which is proved further below.

Theorem 2.3.27. *Let G be a group and $A \in G\text{-}\mathrm{Mod}$. Then there exists a bijection between $H^2(G, A)$ and the set of equivalence classes of group extensions of G by A.*

We first describe the group extension corresponding to the $0 \in H^2(G, A)$. This is given by the *semi-direct product*.

Definition 2.3.28. Let G be a group and $A \in G\text{-}\mathrm{Mod}$. The semi-direct product $A \rtimes G$ is the group with underlying set $A \times G$ and multiplication the *twisted product*

$$(a, g) \cdot (a', g') = (a + g \cdot a', g \cdot g')$$

Example 2.3.29. Let S_n be the symmetric group, whose elements are bijections of the set $[n] = \{1, \ldots, n\}$. The signum of a permutation $\pi \in S_n$ can be defined as the determinant of the corresponding permutation matrix (either 1 or -1). Writing $C_2 = \{1, -1\}$ for the cyclic group, the signum defines a surjective group homomorphism $\mathrm{sgn}(\text{-})\colon S_n \to C_2$ whose kernel is called the alternating group and written A_n. If $n \leq 3$, then A_n is abelian and thus inherits a C_2-action. Let us spell this out for $n = 3$:

We have $S_3 = \{e = 1, (12), (23), (13), (123), (132)\}$ using the cycle notation for permutations (for instance (123) corresponds to the bijection $1 \mapsto 2$, $2 \mapsto 3$ and $3 \mapsto 1$). Then $\mathbb{Z}/3\mathbb{Z} \simeq A_3 = \{e, (123), (132)\} \subset S_3$. The action $C_2 \times A_3 \to A_3$ is given by conjugation with *any* preimage in S_3. For $-1 \in C_2$, a preimage is any transposition, for instance $\mathrm{sgn}((12)) = -1$. Thus

$$-1 \cdot a = (12)a(12)$$

for all $a \in A_3$. Explicitly, this amounts to

$$-1 \cdot (\text{-})\colon A_3 \to A_3, \quad e \mapsto e, \quad (123) \mapsto (12)(123)(12) = (132), \quad (132) \mapsto (123)\,.$$

For $n \geq 4$, the group A_4 is not abelian, and thus there is no canonical action of C_2 on A_4. Nevertheless, there is a split short exact sequence $1 \to A_n \to S_n \to C_2 \to 1$.

For $A \in G\text{-}\mathrm{Mod}$, there is a canonical inclusion $\iota\colon A \to A \rtimes G, a \mapsto (a, 1_G)$, and a canonical surjection $\pi\colon A \rtimes G \to G, (a, g) \mapsto g$ which assemble into a short exact sequence.

As in abelian categories, a group extension

$$0 \to A \to E \xrightarrow{\pi} G \to 1$$

is called split if there exists a section $s\colon G \to E$ of π.

Proposition 2.3.30. *Let G be a group, $A \in G\text{-}\mathrm{Mod}$ and*

$$0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

an extension of G by A. Then the extension is split if and only if it is equivalent to the extension

$$0 \to A \to A \rtimes G \to G \to 1\,.$$

Proof. The extension arising from the semi-direct product is split: the section is given by $s\colon G \to A \rtimes G,\ g \mapsto (0, g)$. For the converse implication, suppose that π admits a section s. We define the map

$$f\colon A \rtimes G \to E, \quad (a, g) \mapsto as(g)\,.$$

This is indeed a group homomorphism: we have

$$\begin{aligned} f(a,g)f(b,h) &= as(g)bs(h) = as(g)bs(g)^{-1}s(g)s(h) \\ &= a(g \cdot b)s(gh) = f(a + (g \cdot b), gh) = f((a,g)(b,h))\,, \end{aligned}$$

where we have used that g acts on b by conjugation with $s(g)$.

The composite $A \subset A \rtimes G \xrightarrow{f}, a \mapsto as(1_G)$ is clearly given by ι. The composite $\pi \circ f$ acts on (a, g) as $\pi(as(g)) = \pi(a)(\pi \circ s(g)) = 1_G g = g$, and is thus given by the epimorphism $A \rtimes G \to G$. This shows that f defines a morphism between extensions of G, A, meaning that the extensions $A \rtimes G$ and E are equivalent. $\square$

The idea of the proof of Theorem 2.3.27 is to pick any section of pointed sets of π and measure its derivations from being a group homomorphism. This derivation will be encoded in the so-called factor set $[\text{-},\text{-}]\colon G \times G \to A$, amounting to a (normalized) 2-cocycle.

Definition 2.3.31. Let G be a group and $A \in G\text{-}\mathrm{Mod}$. Consider a group extension:

$$0 \to A \to E \xrightarrow{\pi} G \to 1$$

Let $s\colon G \to E$ be a section of pointed sets, meaning a map between sets satisfying $\pi \circ s = \mathrm{id}_G$ and $s(1_G) = 1_E$, that is not necessarily a group homomorphism. The associated *factor set* is the function

$$[\text{-},\text{-}]\colon G \times G \to A, \quad (g, h) \mapsto s(g)s(h)s(gh)^{-1}\,.$$

Note that we use $s(g)s(h)s(gh)^{-1} \in \ker(\pi) \simeq A$.

Proposition 2.3.32. *Let G be a group and $A \in G\text{-}\mathrm{Mod}$.*

(1) Consider a group extension

$$0 \to A \to E \xrightarrow{\pi} G \to 1$$

with a section of pointed sets s of π. The section s is a group homomorphism if and only if the factor set $[\text{-},\text{-}]\colon G \times G \to A$ is trivial, meaning constant with value $0 \in A$.

(2) If two extensions of E, E' of G, A with chosen sections of pointed sets have identical factor sets, then the extensions are equivalent.

(3) Any factor set $[\text{-},\text{-}]\colon G \times G \to A$ defines a normalized 2-cocycle, meaning it satisfies

- $[g, 1] = [1, g] = 0$ *for all $g \in G$, and*
- $f[g, h] - [fg, h] + [f, gh] - [f, g] = 0$ *for all $f, g, h \in G$.*

(4) Any normalized 2-cocycle $[\text{-},\text{-}]\colon G \times G \to A$ arises as a factor set of an extension of G by A.

Proof. Part (1) is straightforward: Assume that $[g, h] = 0$ for all $g, h \in G$. Then $1 = s(g)s(h)s(gh)^{-1}$, meaning that $s(g)s(h) = s(gh)$, showing that s is a group homomorphism.

We next show part (2). Consider two extensions E, E' with sections s, s' giving rise to identical factor sets $[\text{-},\text{-}]\colon G \times G \to A$. There are bijections

$$\alpha\colon E \to A \times G, e \mapsto (e\ s(\pi(e))^{-1}, \pi(e))$$

and

$$\beta\colon E' \to A \times G, e \mapsto (e\ s'(\pi'(e))^{-1}, \pi'(e))$$

We claim that $\beta^{-1} \circ \alpha\colon E \to E'$ is a group homomorphism exhibiting the extensions as equivalent. For this, we show that the induced group structures on $A \times G$ coincide. The two arising group structures on $A \times G$ satisfy the following

- $(a, 1) \cdot (b, 1) = (ba, 1)$
- $(a, 1) \cdot (0, g) = (a, g)$
- $(0, g) \cdot (a, 1) = (g \cdot a, a)$
- $(0, g) \cdot (0, h) = ([g, h], gh)$

from which it follows that they coincide (exercise: check this). The first three equalities above are easy to check. For the last equality, note that the inverse α^{-1} of the bijection α is given by $(a, g) \mapsto as(g)$, and thus

$$\begin{aligned} &\alpha(\alpha^{-1}((0, g))\alpha^{-1}((0, h))) = \alpha(s(g)s(h)) \\ &= (s(g)s(h)(s(\pi(s(g)s(h)))^{-1}), \pi(s(g))\pi(s(h))) = ([g, h], gh)\,, \end{aligned}$$

where we used in the last step that π is a group homomorphism and thus $\pi(s(g)s(h)) = \pi(s(g))\pi(s(h)) = gh$. A similar argument applies for β.

For part (3), we note that the associativity of the multiplication in E implies that induced multiplication of $A \times G$ (described above) is also associative. We have

$$\big((0, f)(0, g)\big)(0, h) = ([f, g], fg)(0, h) = ([f, g], 1)(0, fg)(0, h)$$

$$= ([f, g], 1)([fg, h], fgh) = ([f, g], 1)([fg, h], 1)(0, fgh) = ([f, g] + [fg, h], fgh)$$

and

$$(0, f)\big((0, g)(0, h)\big) = (0, f)([g, h], gh) = (0, f)([g, h], 1)(0, gh) = (f[g, h], f)(0, gh)$$

$$= (f[g, h], 1)(0, f)(0, gh) = (f[g, h], 1)([f, gh], fgh) = (f[g, h] + [f, gh], fgh)\,.$$

Thus

$$f[g, h] + [f, gh] = [fg, h] + [f, g]\,.$$

The equation $[g, 1] = [1, g] = 0$ is straightforward to show.

For part (4), we note that given a normalized cocycle $[\text{-},\text{-}]\colon G \times G \to A$, we can equip $A \times G$ with a group structure by using the formulas listed in the proof of part (2). It is not difficult to check that this defined an associative multiplication with unit $(0, 1)$. The inverse of (a, g) is given by $(-g^{-1} \cdot a - g^{-1} \cdot [g, g^{-1}], g^{-1})$. The canonical inclusion $A \subset A \times G$ and projection $A \times G \to G$ then assemble into a SES of groups. $\square$

Remark 2.3.33. Let G be a group. A reduced version $B_*^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$ of the bar construction is as follows: we set

$$B_n^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) = \mathbb{Z}[G] \otimes (\mathbb{Z}[G \backslash \{1_G\}])^{\times n} \otimes \mathbb{Z}$$

where $\mathbb{Z}[G\backslash\{1_G\}] \subset \mathbb{Z}[G]$ is the free abelian group with basis $G\backslash\{1_G\}$. The differential of $B_n^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$ is defined as for $B_n(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$, except that we set to 0 any pure tensor arising in the differential containing an entry 1_G. Then there is a surjection $B_n^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \to B_n(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$, mapping a pure tensor $r_0 \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y)$ to itself when $r_i \neq 1_G$ for all $i \in [0, n]$ and to 0 otherwise. This surjection extends to a chain map

$$B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \to B_*^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \,.$$

The proof Lemma 2.3.18 applies verbatim to show that $B_*^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \to \mathbb{Z}$ is a projective resolution.

It follows that the inclusion $B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}) \to B_*^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$ is a chain homotopy equivalence. We can identify the 2-cocycles of $\operatorname{Hom}_{\mathbb{Z}[G]}(B_*^{\text{red}}(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}), A)$ under the inclusion into the 2-cocyles of $\operatorname{Hom}_{\mathbb{Z}[G]}(B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z}), A)$ with the normalized 2-cocycles $[\text{-}, \text{-}] \colon G \times G \to A$, meaning maps that satisfy $[1, g] = [g, 1] = 0$.

Lemma 2.3.34. *Let G be a group and $A \in G$-*Mod*. Consider an extension $0 \to A \to E \xrightarrow{\pi} G \to 1$ with factor set $[\text{-}, \text{-}] \colon G \times G \to A$ arising from a pointed section s of π. Then a factor set $[\text{-}, \text{-}]' \colon G \times G \to A$ differs from $[\text{-}, \text{-}]$ by a normalized coboundary $d(\beta)$ with $\beta \colon G \to A$ satisfying $\beta(1_G) = 0$, meaning that*

$$[g, h]' - [g, h] = \beta(g) - \beta(gh) + g \cdot \beta(h) \,,$$

if and only if $[\text{-}, \text{-}]'$ arises as the factor set of a possibly different choice of pointed section s' of π.

Proof. Suppose that $[\text{-}, \text{-}], [\text{-}, \text{-}]'$ differ by a coboundary $d(\beta)$. Then define $s' \colon G \to E$ by $s'(g) = \beta(g)s(g)$. Note that $\pi \circ s' = \mathrm{id}_G$ since $\beta(g) \in A \subset E$. Then the factor set $[g, h]_{s'}$ arising from s' is given by

$$\begin{aligned}
[g, h]_{s'} &= s'(g)s'(h)s'(gh)^{-1} \\
&= \beta(g)s(g)\beta(h)s(h)s(gh)^{-1}\beta(gh)^{-1} \\
&= \beta(g) + s(g)\beta(h)s(g)^{-1} + s(g)s(h)s(gh)^{-1} - \beta(gh) \\
&= [g, h] + \beta(g) - \beta(gh) + g \cdot \beta(h) \,,
\end{aligned}$$

meaning that $[g, h]_{s'} = [g, h]'$.

Conversely, given a pointed section $s' \colon G \to E$, we note that there is a pointed morphism $\beta \colon G \to A$ with $\beta(g) = s'(g)s(g)^{-1}$. The computation above shows that the factor set $[\text{-}, \text{-}]_{s'}$ arising from s' differes from $[\text{-}, \text{-}]$ by the boundary of β: $[g, h]_{s'} = [g, h] + \beta(g) - \beta(gh) + g \cdot \beta(h)$. $\square$

Proof of Theorem 2.3.27. Consider the set $\mathbb{E} = \{0 \to A \to E \xrightarrow{\pi} G \to 1\}/\sim$ of equivalence classes of extensions of G by A. We define a morphism

$$\phi\colon \mathbb{E} \longrightarrow H^2(G, A), E \mapsto [\text{-}, \text{-}]$$

mapping each equivalence class of extension to the factor set of a choice of based section of π of a choice of representative $0 \to A \to E \xrightarrow{\pi} X \to 1$ of the extension. That the factor set defines a normalized 2-cocycle follows from Proposition 2.3.32.(3) and Remark 2.3.33. The well-definedness of this morphism follows from Lemma 2.3.34. The surjectivity of ϕ follows from part (4) of Proposition 2.3.32. The injectivity of ϕ follows from Lemma 2.3.34. $\square$

2.4 Quiver representations

In this chapter, we will introduce quivers and their representation categories. These form abelian categories in which the projective and injective objects can be explicitly described. Many interesting abelian categories turn out to be equivalent to representation categories of quivers, so that methods developed for their study can be productively applied in many situations. The study of quivers and their representations is a subfield of the (large and diverse) field of *representation theory*.

2.4.1 Quivers and path algebras

Definition 2.4.1. A quiver $Q = (Q_0, Q_1, s, t)$ consists of two finite sets Q_0 of vertices and Q_1 of arrows, together with two maps $s, t\colon Q_1 \to Q_0$, called source and target maps.
If $a \in Q_1$ is an arrow with $s(a) = i$ and $t(a) = j$, we write $a\colon i \to j$.

We can consider a quiver as a finite directed graph. Note that loops (meaning arrows l satisfying $s(l) = t(l)$), as well as multiple arrows between two given vertices, are allowed.

Example 2.4.2. An example of a quiver with four vertices and a loop:

$$\begin{tikzcd}
 & 3 & \\
1 \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[u, shift left] \arrow[u, shift right] \arrow[loop below] & 4
\end{tikzcd}$$

Quiver representations are most often studied over fields, since these are better behaved. Considering representations over rings is of course possible. We fix a field k. We will denote the category LMod_k by Vect_k.

Definition 2.4.3. Let Q be a quiver.

(1) A (k-linear) representation V of Q consists of an assignment of

- a k-vector spaces V_i to each vertex $i \in Q_0$,
- a linear map $V_a \colon V_i \to V_j$ to each arrow $a \colon i \to j$.

A representation V of Q is called finite dimensional if the vector space V_i is finite dimensional for all $i \in Q_0$.

(2) A morphism between representation $f \colon V \to W$ of Q consists of a collection of linear maps $f_i \colon V_i \to W_i$, $i \in Q_0$, such that for all $a \in Q_1$ the diagram

$$\begin{tikzcd} V_{s(a)} \arrow[r, "V_a"] \arrow[d, "f_{s(a)}"'] & V_{t(a)} \arrow[d, "f_{t(a)}"] \\ W_{s(a)} \arrow[r, "W_a"] & W_{t(a)} \end{tikzcd}$$

commutes.

We denote by $\mathrm{Rep}_k(Q)$ the category of representations of Q.

Definition 2.4.4. Let Q be a quiver and label its vertices by $Q_0 = \{1, \dots, m\}$. Let $V \in \mathrm{Rep}_k(Q)$. The dimension vector of V is

$$\underline{\dim}(V) = (\dim_k V_1, \dots, \dim_k V_m) \in \mathbb{N}^m .$$

Example 2.4.5. The following quiver is called the A_2 quiver:

$$A_2 = \qquad 1 \xrightarrow{a} 2$$

A representation of A_2 consists of two vector spaces and a linear map:

$$V_1 \xrightarrow{V_a} V_2$$

A morphism f between representations of A_2 amounts to a commutative square of the following form:

$$\begin{tikzcd} V_1 \arrow[r, "V_a"] \arrow[d, "f_1"] & V_2 \arrow[d, "f_2"] \\ W_1 \arrow[r, "W_a"] & W_2 \end{tikzcd}$$

Given a representation V of A_2, the linear map V_a factors as $V_1 \twoheadrightarrow \mathrm{Im}(V_a) \hookrightarrow V_2$. We further have $V_1 \simeq \ker(V_a) \oplus \mathrm{Im}(V_a)$ and $V_2 \simeq \mathrm{Im}(V_a) \oplus \mathrm{coker}(V_a)$. Any representation of A_2 is thus isomorphic in $\mathrm{Rep}(A_2)$ to a representation of the form

$$U_1 \oplus U_2 \xrightarrow{\begin{pmatrix} 0 & \mathrm{id}_{U_2} \\ 0 & 0 \end{pmatrix}} U_2 \oplus U_3 \,, \tag{31}$$

with $U_1, U_2, U_3 \in \mathrm{Vect}_k$. Direct sums of quiver representations are computed 'point-wise' at the vertices of the quiver. The representation (31) is thus the direct sum of the following three representations:

$$U_1 \longrightarrow 0$$

$$U_2 \xrightarrow{\mathrm{id}_{U_2}} U_2$$

$$0 \longrightarrow U_3$$

The dimension vectors of the three above representations are

$$(\dim_k(U_1), 0), \quad (\dim_k(U_2), \dim_k(U_2)), \quad (0, \dim_k(U_3))\,.$$

Example 2.4.6. A representation of the one-loop quiver

$$Q = \begin{tikzcd} 1 \arrow[loop above, "l"] \end{tikzcd}$$

amounts to a vector space V together with an endomorphism $f\colon V \to V$. Suppose that V is finite dimensional of dimension n. Upon choosing a basis of V, we can represent f by an $n \times n$-matrix M_f. When changing the basis, the matrix M_f changes by conjugation $T^{-1}M_fT$.
Equivalence classes of finite dimensional representations of Q are therefore in bijection with square matrices up to conjugation. If k is algebraically closed, the latter can be classified using Jordan normal forms.

Given a quiver Q, it is an interesting question whether one can classify the finite-dimensional representations of Q. This is however for most quivers not possible.

Remark 2.4.7. Let Q be a quiver and $V, W \in \mathrm{Rep}_k(Q)$. The abelian group $\mathrm{Hom}_{\mathrm{Rep}_k(Q)}(V, W)$ of morphisms of quiver representations canonically inherits the structure on a k-vector space: given $\lambda \in k$, the action on a morphism $f\colon V \to W$ is given by the morphism $\lambda \cdot f\colon V \to W$ with $(\lambda \cdot f)_i(v_i) = \lambda f_i(v_i) \in W_i$ for all $i \in Q_0$ and $v_i \in V_i$. Composition of morphisms

$$\mathrm{Hom}_{\mathrm{Rep}_k(Q)}(U, V) \times \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(V, W) \to \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(U, W)$$

in $\mathrm{Rep}_k(Q)$ is further k-bilinear. Such a structure is also called a Vect_k-*enrichment* of the category $\mathrm{Rep}_k(Q)$.

We next describe the category $\mathrm{Rep}_k(Q)$ as the module category of the path algebra of Q. For context, we first recall the notion of a k-algebra:

Definition 2.4.8. Let k be a field. A k-algebra A consists of a k-vector space A together with a k-bilinear morphism $\mu\colon A \times A \to A$ which equips A with the structure of a (unital and associative) ring.

Example 2.4.9. Given a group G and a field k, the group ring $k[G]$ from Definition 2.3.6 is canonically a k-algebra. Thus, $k[G]$ is also called the *group algebra.*

Remark 2.4.10. If A is a k-algebra, there is a morphism of rings $k \to A$, mapping $\lambda \in k$ to $\lambda \cdot 1_A \in A$. We can thus consider A as an A-k-bimodule. We thus obtain a functor

$$\mathrm{Hom}_A(A, \text{-})\colon \mathrm{LMod}_A \to \mathrm{Vect}_k$$

assigning to each A-module its *underlying k-vector space.*
Given a k-algebra A, we denote by $\mathrm{LMod}_A^{\mathrm{fd}} \subset \mathrm{LMod}_A$ the full subcategory whose objects are A-modules whose underlying k-vector space is finite dimensional.

Definition 2.4.11. (1) Let Q be a quiver. A path in Q of length $m \geq 0$ consists of a finite collection of vertices $v_0, \ldots, v_m$ and arrows $a_i\colon v_i \to v_{i+1}$, $i \in [0, m-1]$. Given such a path p, we set $s(p) = v_0$ and $t(p) = v_m$.

(2) The lazy path e_v at a vertex v is defined as the path consisting of the vertex $v = v_0$ and no arrows.

(3) Any non-lazy path is determined by the arrows contained in it, which satisfy $s(a_{i+1}) = t(a_i)$. We may thus write paths as $p = a_m a_{m-1} \ldots a_1$, or $p = e_v$.

Definition 2.4.12. Let Q be a quiver. The path algebra kQ is the k-algebra with

- the basis of the k-vector space kQ given by the set of paths in Q,
- and multiplication the unique bilinear map determined on basis elements by

$$(a_m \ldots a_1) \cdot (b_n \ldots b_1) = \begin{cases} a_m \ldots a_1 b_n \ldots b_1 & s(a_1) = t(b_n) \\ 0 & 0\,. \end{cases}$$

  The product of two paths is thus their composite if it makes sense to compose them, and it is zero otherwise.

One readily checks that the above defined multiplication is associative. The unit 1_{kQ} of the path algebra is given by the sum of the lazy paths $\sum_{v \in Q_0} e_v$.

Lemma 2.4.13. *Let Q be a quiver and M a kQ-module. For $i \in Q_0$ consider the k-vector space*

$$e_i M = \{e_i \cdot m | m \in M\} \subset M\,.$$

Then the inclusions $e_i M \subset M$ exhibit the k-vector space underlying M as the direct sum

$$M \simeq \bigoplus_{i \in Q_0} e_i M\,.$$

Proof. We must show that we can write each element $m \in M$ uniquely as a sum $m = \sum_{i \in Q_i} x_i$ of elements $x_i \in e_i M$ (Exercise: convince yourself that this characterizes the direct sum). Firstly, we have $1_{kQ} = \sum_{i \in Q_0} e_i$, and so $m = 1_{kQ} m = \sum_{i \in Q_0} e_i m$, where $e_i m \in e_i M$. To show uniqueness, suppose that $\sum_{i\in Q_i} x_i = \sum_{i\in Q_i} y_i$ with $x_i, y_i \in e_i M$. Let $j \in Q_0$. Since $e_j \cdot e_i = 0$ for $j \neq i$ and $e_j^2 = e_j$, we find that $e_j \cdot m_i = 0$ if $m_i \in e_i M$ and $i \neq j$, and $e_j \cdot m_j = m_j$ if $m_j \in e_j M$. By multiplying with e_j, we thus obtain the desired equality:

$$x_j = e_j x_j = e_j \sum_{i\in Q_i} x_i = e_j \sum_{i\in Q_i} y_i = e_j y_j = y_j\,.$$

□

Proposition 2.4.14. *There exists an equivalence of categories*

$$\mathrm{Rep}_k(Q) \simeq \mathrm{LMod}_{kQ}\;.$$

In particular, $\mathrm{Rep}_k(Q)$ *is an abelian category with enough projectives and injectives.*

Proof. Let V be a representation of Q. We define a kQ-module structure on $M := \bigoplus_{i\in Q_0} V_i$ as follows. Let m be the number of vertices of Q. Given $(v_1,\dots,v_m) \in \bigoplus_{i\in Q_0} V_i = M$, we define the action of a path $p = a_m \dots a_1$ with $s(p) = j$ and $t(p) = l$ as follows:

$$p\cdot(v_1,\dots,v_m) = (0,\dots,0,\underbrace{V_{a_m}\circ\dots\circ V_{a_1}(v_j)}_{\in V_l},0,\dots,0)\,.$$

For a lazy path, we set $e_i(v_1,\dots,v_m) = (0,\dots,v_i,\dots 0)$. This extends to a bilinear action $kQ \times M \to kQ$ via

$$\left(\sum_{i=1}^n \lambda_i p_i\right)\cdot(v_1,\dots,v_m) = \sum_{i=1}^n \lambda_i(p_i\cdot(v_1,\dots,v_m))\,,$$

for p_i a path and $\lambda_i \in k$. The assignment $V \mapsto M = \bigoplus_{i\in Q_0} V_i$ extends to a functor

$$F\colon\ \mathrm{Rep}_k(Q) \to \mathrm{LMod}_{kQ}$$

assigning to a morphism $f\colon V \to W$ the morphism $\bigoplus_{i\in Q_0} f_i\colon \bigoplus_{i\in Q_0} V_i \to \bigoplus_{i\in Q_0} W_i$. One readily checks that $\bigoplus_{i\in Q_0} f_i$ is indeed a morphism of kQ-modules.

Given a kQ-module M and $i \in Q_0$, consider the k-vector space $V_i = e_i M$. For $(a\colon i \to j) \in Q_1$, we obtain a linear map $V_a\colon V_i \to V_j$ by setting $V_a(e_i\cdot m) = a\cdot e_i\cdot m = a\cdot m$. This defines a representation V of Q. Given a morphism of kQ-modules $f\colon M \to N$, we have that $f(e_i\cdot m) = e_i\cdot f(m)$, and so f restricts to a morphism $f_i\colon V_i = e_i M \to e_i N = W_i$, which one finds to define a morphism of Q-representations. We thus obtain a functor

$$G\colon\ \mathrm{LMod}_{kQ} \to \mathrm{Rep}_k(Q),\quad M \mapsto V\,.$$

We note that $F\circ G$ is not identical to $\mathrm{id}_{\mathrm{LMod}_{kQ}}$, since the direct sum $\bigoplus_{i\in Q_0} V_i$ is only determined up to isomorphism. Hence, F and G are not inverse isomorphisms of categories. There are nevertheless canonical natural isomorphisms $F\circ G \simeq \mathrm{id}_{\mathrm{LMod}_{kQ}}$ and $G\circ F \simeq \mathrm{id}_{\mathrm{Rep}_k(Q)}$, where the former uses Lemma 2.4.13. We thus find that F, G are equivalences of categories in the sense of Remark 1.6.33. □

Remark 2.4.15. Let Q be a quiver and $i \in Q_0$. Then $kQe_i \subset kQ$ is the left ideal of k-linear sums of paths which start at i. As kQ-modules, there exists an equivalence $kQ \simeq kQe_i \oplus kQ(1 - e_i)$, where $kQ(1 - e_i)$ consists of sums of paths which do not start at e_i. We thus see that kQe_i is projective. We will denote $P_i \coloneqq kQe_i$. Note that

$$\bigoplus_{i \in Q_0} P_i \simeq kQ \in \mathrm{LMod}_{kQ} \ .$$

Example 2.4.16. The D_4-quiver is the following quiver:

$$D_4 = \quad \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a_1"] & 2 \arrow[ur, "a_2"] \arrow[dr, "a_3"] & \\ & & 4 \end{tikzcd}$$

The four projective kD_4 modules $P_1, \dots, P_4$ correspond to the following representations

$$P_1 = \quad \begin{tikzcd} & & k \\ k \arrow[r, "\mathrm{id}"] & k \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \\ & & k \end{tikzcd}$$

$$P_2 = \quad \begin{tikzcd} & & k \\ 0 \arrow[r] & k \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \\ & & k \end{tikzcd}$$

$$P_3 = \quad \begin{tikzcd} & & k \\ 0 \arrow[r] & 0 \arrow[ur] \arrow[dr] & \\ & & 0 \end{tikzcd}$$

$$P_4 = \quad \begin{tikzcd} & & 0 \\ 0 \arrow[r] & 0 \arrow[ur] \arrow[dr] & \\ & & k \end{tikzcd}$$

Lemma 2.4.17. *Let Q be a quiver, $i \in Q_0$, and $V \in \mathrm{Rep}_k(Q)$. Then*

$$\mathrm{Hom}_{\mathrm{Rep}_k(Q)}(P_i, V) \simeq V_i \in \mathrm{Vect}_k \ .$$

Proof. Let $M = \bigoplus_{i \in Q_0} V_i$ be the kQ-module corresponding to V. We have

$$\mathrm{Hom}_{kQ}(kQ, M) \simeq M$$

as k-vector spaces. The subspace $V_i = e_i M \subset M$ corresponds under this isomorphism to the subspace of morphisms $kQ \to M$, which map 1_{kQ} to $V_i = e_i M \subset M$. Let $i \neq j \in Q_0$. Let $f \colon kQ \to M$ with $f(1_{kQ}) = e_i m \in e_i M$ for some $m \in M$.

The composite $P_j \to kQ \to M$ must map any path $p = p1_{kQ}$ with $s(p) = j$ to $p \cdot f(1_{kQ}) = p \cdot e_i \cdot m = 0 \cdot m = 0$. Thus, the restriction of f to any P_j with $j \neq i$ vanishes. Using the decomposition $kQ \simeq \bigoplus_{j \in Q_0} P_j$, we find that any morphism $f \colon kQ \to M$ with $f(1_{kQ}) \in V_i$ is uniquely determined by its restriction to P_i. We thus obtain the desired isomorphism

$$\mathrm{Hom}_{kQ}(P_i, M) \simeq \{f \in \mathrm{Hom}_{kQ}(kQ, M) | f(1_{kQ}) \in V_i\} = e_i M = V_i \, .$$

□

We can also associate an injective object $I_i \in \mathrm{LMod}_{kQ}$ with each vertex i of Q. To describe the injectives, we use a duality involving the opposite quiver.

Definition 2.4.18. Given a quiver Q, we let Q^{op} be the opposite quiver with $Q_0^{\mathrm{op}} = Q_0$, $Q_1^{\mathrm{op}} = Q_1$, $s_{Q^{\mathrm{op}}}(a) = t_Q(a)$, and $t_{Q^{\mathrm{op}}}(a) = s_Q(a)$ for all $a \in Q_1$. Informally speaking, Q^{op} is obtained from Q by reversing all the arrows.

We note that $(kQ)^{\mathrm{op}} \simeq k(Q^{\mathrm{op}})$.

Proposition 2.4.19. *There exist equivalences of categories*

$$\mathrm{LMod}_{kQ}^{\mathrm{fd}} \simeq (\mathrm{RMod}_{kQ}^{\mathrm{fd}})^{\mathrm{op}} \simeq (\mathrm{LMod}_{kQ^{\mathrm{op}}}^{\mathrm{fd}})^{\mathrm{op}} \, .$$

Proof sketch. The equivalence of categories $\mathrm{LMod}_{kQ}^{\mathrm{fd}} \simeq (\mathrm{RMod}_{kQ}^{\mathrm{fd}})^{\mathrm{op}}$ is given by the assignment

$$M \mapsto M^* = \mathrm{Hom}_{\mathrm{Vect}_k}(U(M), k) \, ,$$

where $U(M)$ is the vector space underlying M. Note that $U(M^*)$ is the dual vector space of $U(M)$. Further, M^* is a right kQ-module via the action $(f \cdot p)(m) = f(p \cdot m)$ for $p \in kQ$, $m \in M$ and $f \colon M \to k$. To show that the functor (-)* is an equivalence of categories, it suffices to note that the double dual $(M^*)^*$ is isomorphic to M (since $U(M)$ is finite dimensional), and this isomorphism is natural in M.

The second equivalence of categories follows from $(kQ)^{\mathrm{op}} \simeq k(Q^{\mathrm{op}})$. □

A consequence of Proposition 2.4.19 is that finite-dimensional projective kQ^{op}-modules are in bijection with finite-dimensional injective kQ-modules. We thus obtain for each vertex $i \in Q_0 = Q_0^{\mathrm{op}}$ an injective kQ-module I_i whose underlying vector space has a basis indexed by the set of paths in Q ending in i.

Example 2.4.20. Continuing Example 2.4.16, we find the following four injective D_4-modules:

$$I_1 = \begin{tikzcd} & & 0 \\ k \arrow[r] & 0 \arrow[ur] \arrow[dr] & \\ & & 0 \end{tikzcd}$$

$$I_2 = \begin{tikzcd} & & 0 \\ k \arrow[r, "\mathrm{id}"] & k \arrow[ur] \arrow[dr] & \\ & & 0 \end{tikzcd}$$

$$
\begin{tikzcd}[row sep=small]
 & & k \\
I_3 = \quad k \arrow[r, "\mathrm{id}"] & k \arrow[ur, "\mathrm{id}"] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}[row sep=small]
 & & 0 \\
I_4 = \quad k \arrow[r, "\mathrm{id}"] & k \arrow[ur] \arrow[dr, "\mathrm{id}"'] & \\
 & & k
\end{tikzcd}
$$

2.4.2 Extensions and the Euler form

Given a kQ-module M, and $i \in Q_0$, recall that e_iM is the k-vector space assigned by the corresponding Q-representation to the vertex i.

Proposition 2.4.21. *Let Q be a quiver and $M \in \mathrm{LMod}_{kQ}$. There is a chain complex in LMod_{kQ}*

$$\tilde{M}_* = \quad \cdots \to 0 \to \bigoplus_{a \in Q_1} P_{t(a)} \otimes_k e_{s(a)}M \xrightarrow{f} \bigoplus_{i \in Q_0} P_i \otimes_k e_iM$$

with f acting on a pure tensor $p \otimes_k m \in P_{t(a)} \otimes_k e_{s(a)}M$, as

$$f(p \otimes_k m) = pa \otimes_k m - p \otimes_k am\,.$$

There is further an augmentation map

$$\epsilon \colon \bigoplus_{i \in Q_0} P_i \otimes_k e_iM \to M$$

acting on a pure tensor $p \otimes_k m \in P_i \otimes_k e_iM$ as

$$\epsilon(p \otimes_k m) = p \cdot m\,,$$

such that $\epsilon \colon \tilde{M}_* \to M$ is a projective resolution.

The chain complex $\tilde{M}_*$ is called the standard resolution of M.

Proof. The construction of the morphisms f and ϵ uses the k-bilinar multiplication map $kQ \times kQ \to kQ$ and the k-bilinear action $kQ \times M \to M$, as well as the universal property of the k-linear tensor product, and is left to the reader.

Clearly, we have $\epsilon \circ f = 0$. We now prove that the sequence

$$0 \to \bigoplus_{a \in Q_1} P_{t(a)} \otimes_k e_{s(a)}M \xrightarrow{f} \bigoplus_{i \in Q_0} P_i \otimes_k e_iM \xrightarrow{\epsilon} M \to 0$$

is exact. We have $\epsilon(e_i \otimes_k m) = m$ for all $m \in e_iM \subset M$, so that ϵ is surjective, meaning an epimorphism.

We next show that f is a monomorphism. Let $0 \neq X \in \ker(f)$. We can write $X = \sum_{a \in Q_1} \sum_{p, s(p) = t(a)} p \otimes_k x_{p,a}$, where the second sum runs over all paths p starting at $t(a)$, $x_{p,a} \in e_{s(a)}M$, and all but finitely many $x_{p,a}$ vanish. For a given $a \in Q_1$, let q_a be the path of maximal length with source $t(a)$ and $x_{q,a} \neq 0$. Then

$$0 = f(X) = \sum_{a\in Q_1} \sum_{p,s(p)=t(a)} pa \otimes_k x_{p,a} - \sum_{a\in Q_1} \sum_{p,s(p)=t(a)} p \otimes_k ax_{p,a} \,.$$

For any $a \in Q_1$, inspecting the $q_a a$ component, the only tensors with the longest path $q_a a$ in the first component is in the term $q_a a \otimes_k x_{p,a}$, which implies that $x_{p,a} = 0$. This is a contradiction. Thus $X = 0$.

Finally, we show that $\mathrm{Im}(f) = \ker(\epsilon)$. We can write $X \in \bigoplus_{i\in Q_0} P_i \otimes_k e_i M$ as $X = \sum_{i\in Q_0} \sum_{p,s(p)=i} p \otimes_k x_p$ where the second sum runs over paths starting at i and has finitely many non-zero terms. Suppose that $\epsilon(X) = 0$. Let $p = a_l \cdots a_1$ be a path of maximal length l such that $x_p \neq 0$. Note that there are only finitely many, say n, such paths. After subtracting $f((a_l \cdots a_2) \otimes_k x_p) = p \otimes_k x_p - (a_l \cdots a_2) \otimes_k a_1 \cdot x_p$ from X, we find that $n-1$ paths of length l remain. Repeating this step $n-1$ times, we find $Y \in \mathrm{Im}(f)$ such that the longest path appearing in $X - Y$ has length $l-1$. Repeating this argument $l-2$-many times, we find that $X = \tilde{X} + \tilde{Y}$ with $\tilde{Y} \in \mathrm{Im}(f)$ and $\tilde{X} = \sum_{i\in Q_0} e_i \otimes_k \tilde{x}_i$. We then have

$$\sum_{i\in Q_0} \tilde{x}_i = \epsilon(\tilde{X}) \overset{\epsilon\circ f=0}{=} \epsilon(X) = 0 \,,$$

and thus $\tilde{x}_i = 0$ for all $i \in I$ by Lemma 2.4.13. Hence $\tilde{X} = 0$ and $X = \tilde{Y} \in \mathrm{Im}(f)$. □

Definition 2.4.22. Let Q be a quiver and $V, W \in \mathrm{Rep}_k(Q)$ be two finite-dimensional representations. We set

$$\chi_Q(V, W) := \dim_k \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(V, W) - \dim_k \mathrm{Ext}^1_{\mathrm{Rep}_k(Q)}(V, W) \,.$$

For the above definition, we note that since the Hom-functor

$$\mathrm{Hom}_{\mathrm{Rep}_k(Q)}(\text{-}, N) \colon \mathrm{Rep}_k(Q)^{\mathrm{op}} \to \mathrm{Ab}$$

factors through the forgetful functor $\mathrm{Vect}_k \to \mathrm{Ab}$, see Remark 2.4.7, so does its right derived functor $\mathrm{Ext}^1_{\mathrm{Rep}_k(Q)}(\text{-}, N)$.

Definition 2.4.23 (Euler form)**.** Let Q be a quiver with $Q_0 = \{1, \ldots, m\}$. The Euler form of Q is the bilinear map

$$\langle \text{-}, \text{-} \rangle_Q \colon \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$$

defined by

$$\langle \underline{u}, \underline{v} \rangle_Q = \sum_{i\in Q_0} u_i \cdot v_i - \sum_{a\in Q_1} u_{s(a)} v_{t(a)} \,.$$

We note that, in general, the Euler form is not symmetric. We can describe the characteristic from Definition 2.4.22 in terms of the Euler form of the dimension vectors from Definition 2.4.4.

Theorem 2.4.24. *Let Q be a quiver with $Q_0 = \{1, \dots, m\}$. Let $V, W \in \mathrm{Rep}_k(Q)$ be two finite-dimensional representations. Then*

$$\chi_Q(V, W) = \langle \underline{\dim}(V), \underline{\dim}(W) \rangle_Q \,.$$

Proof. Consider the short exact sequence

$$0 \to \bigoplus_{a \in Q_1} P_{t(a)} \otimes_k V_{s(a)} \xrightarrow{f} \bigoplus_{i \in Q_0} P_i \otimes_k V_i \xrightarrow{\epsilon} V \to 0 \,.$$

Using that the first two terms are projective, the LES of the derived functor of $\mathrm{Hom}_{\mathrm{Rep}_k(Q)}(\text{-}, W) \colon \mathrm{Rep}_k(Q) \to \mathrm{Vect}_k$ reads as

$$\begin{aligned} C_* = \quad & 0 \to \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(V, W) \to \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(\bigoplus_{i \in Q_0} P_i \otimes_k V_i, W) \\ & \to \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(\bigoplus_{a \in Q_1} P_{t(a)} \otimes_k V_{s(a)}, W) \to \mathrm{Ext}^1_{\mathrm{Rep}_k(Q)}(V, W) \to 0 \,. \end{aligned}$$

Since $P_i \otimes_k V_i \simeq \bigoplus_{\dim_k(V_i)} P_i$ and $P_{t(a)} \otimes_k V_{s(a)} \simeq \bigoplus_{\dim_k(V_{s(a)})} P_{t(a)}$ as Q-representations, we have for $i \in Q_0$

$$\begin{aligned} \dim_k \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(P_i \otimes_k V_i, W) &= \dim_k(V_i) \cdot \dim_k \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(P_i, W) \\ &\overset{\textit{Lemma } 2.4.17}{=} \dim_k(V_i) \cdot \dim_k(W_i) \end{aligned}$$

and similarly for $a \in Q_1$

$$\dim_k \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(P_{t(a)} \otimes_k V_{s(a)}, W) = \dim_k V_{s(a)} \cdot \dim_k W_{t(a)} \,.$$

The exactness of the above LES C_* implies that its Euler characteristic $\sum_{i \in \mathbb{Z}} (-1)^i \dim_k(C_i)$ vanishes ('homology commutes with Euler characteristic', see Sheet 5, Exercise 3). Thus

$$\begin{aligned} \chi_Q(V, W) = \dim_k \mathrm{Hom}_{\mathrm{Rep}_k(Q)}(V, W) - \dim_k \mathrm{Ext}^1_{\mathrm{Rep}_k(Q)}(V, W) = \\ = \sum_{i \in Q_0} \dim_k(V_i) \cdot \dim_k(W_i) - \sum_{a \in Q_1} \dim_k V_{s(a)} \cdot \dim_k W_{t(a)} = \langle V, W \rangle_Q \,, \end{aligned}$$

as desired. $\square$

2.4.3 Simple modules

Definition 2.4.25. Let $\mathcal{C}$ be an abelian category.

(1) A non-zero object $S \in \mathcal{C}$ is called simple if for any monomorpism $i \colon X \to S$ either $X = 0$ or i is an isomorphism.

(2) A non-zero object $X \in \mathcal{C}$ is called indecomposable if $X \simeq X_1 \oplus X_2$ implies that $X_1 \simeq 0$ or $X_2 \simeq 0$. We note that simple objects are indecomposable.

Let $S \in \mathcal{C}$ be simple. Given an epimorphism $p\colon S \to Y$, we can consider the corresponding short exact sequence $0 \to X \xrightarrow{i} S \xrightarrow{p} Y \to 0$ with $X = \ker(p)$. Then either $X = 0$ or i is an isomorphism. Since $Y = \operatorname{coker}(i)$, in the first case, we find that p is an isomorphism. In the second case, we find that $Y = 0$. We thus see:

Lemma 2.4.26. *An object $S \in \mathcal{C}$ is simple if and only if for any epimorphism $p\colon S \to Y$ either $Y = 0$ or p is an isomorphism.*

Example 2.4.27. (1) The 1-dimensional vector space $k \in \mathrm{Vect}_k$ is simple and indecomposable. Up to isomorphism, it is the only indecomposable or simple object of Vect_k.

(2) The abelian group $\mathbb{Z} \in \mathrm{Ab}$ is indecomposable but not simple.

Definition 2.4.28. Let Q be a quiver and $i \in Q_0$. We define the i-th simple $S_i \in \mathrm{Rep}_k(Q)$ as the unique representation with

$$(S_i)_j = \begin{cases} k & j = i\,, \\ 0 & \text{else}\,. \end{cases}$$

Example 2.4.29. Let $Q = A_2$ be the A_2-quiver. Then the two simple representations of Q are

$$S_1 = (k \longrightarrow 0)$$

and

$$S_2 = (0 \longrightarrow k)\,.$$

Lemma 2.4.30. *Let Q be a quiver and $i, j \in Q$. Then*

$$\mathrm{Ext}^n_{\mathrm{LMod}_{kQ}}(P_i, S_j) \simeq \begin{cases} k & i = j, n = 0 \\ 0 & \textit{else.} \end{cases}$$

Proof. The description of $\mathrm{Ext}^0_{\mathrm{LMod}_{kQ}}(P_i, S_j) \simeq \mathrm{Hom}_{\mathrm{LMod}_{kQ}}(P_i, S_j)$ follows from Lemma 2.4.17. The vanishing of the higher Ext groups follows from the fact that P_i is projective. $\square$

Proposition 2.4.31. *Let Q be a quiver and $i, j \in Q_0$ with $i \neq j$. Then*

$$\dim_k \mathrm{Ext}^1_{\mathrm{LMod}_{kQ}}(S_i, S_j) = \#\textit{arrows } a\colon i \to j\,.$$

Proof. The standard resolution of the simple S_i yields the following short exact sequence

$$0 \to \bigoplus_{a \in Q_1, s(a) = i} P_{t(a)} \to P_i \to S_i \to 0\,.$$

Applying $\mathrm{Hom}_{kQ}(\text{-}, S_j)$, the associated LES is given as follows:

$$\begin{aligned} \ldots \to \underbrace{\mathrm{Ext}^1_{kQ}(P_i, S_j)}_{\simeq 0} &\to \mathrm{Ext}^1_{kQ}(S_i, S_j) \to \mathrm{Hom}_{kQ}(\bigoplus_{a \in Q_1, s(a) = i} P_{t(a)}, S_j) \to \\ \to \underbrace{\mathrm{Hom}_{kQ}(P_i, S_j)}_{\simeq 0} &\to \mathrm{Hom}_{kQ}(S_i, S_j) \to 0 \end{aligned}$$

Thus $\mathrm{Ext}^1_{kQ}(S_i, S_j) \simeq \mathrm{Hom}_{kQ}(\bigoplus_{a \in Q_1, s(a) = i} P_{t(a)}, S_j)$. Using that $\mathrm{Hom}_{kQ}(P_{t(a)}, S_j) \simeq k$ if $t(a) = j$ and $\mathrm{Hom}_{kQ}(P_{t(a)}, S_j) \simeq 0$ if $t(a) \neq j$, the assertion follows. $\square$

3 The derived category

The contents of this section will not be part of either exam.

A common slogan in homological algebra goes as follows:

Chain complexes are good, remembering just their homology is bad.

Chain complexes allow us to consider chain maps between them, which then leads for instance to long exact sequences. Most maps between homology groups are induced by chain maps, and would actually be difficult to construct directly from just the homology groups. The passage from a chain complex to its homology loses information.

The derived functors L_iF of a right exact functor $F\colon \mathcal{A} \to \mathcal{B}$ were defined as individual homology groups. According to the above slogan, it is desirable to assemble all the derived functors into a single chain complex valued derived functor L_*F. Trying to construct L_*F as a functor $\mathcal{A} \to \mathrm{Ch}(\mathcal{B})$ is doomed to fail: the chain complex describing L_*F is only well-defined up to quasi-isomorphism (in fact even up to the more restrictive chain homotopy).

We will thus construct a category, called the *derived category* $\mathcal{D}(\mathcal{B})$ of the abelian category $\mathcal{B}$, whose objects are such chain complexes up to quasi-isomorphism. The derived functors of F will then assemble into a functor $\mathcal{A} \to \mathcal{D}(\mathcal{B})$. Under further assumptions, we will be able to extend this to a functor between the derived categories. If F is the left adjoint of G, we can then further ask if we get a 'derived adjunction' $L_*F\colon \mathcal{D}(A) \longleftrightarrow \mathcal{D}(B) : R^*G$.

There are many further motivations to study derived categories of abelian categories. Derived categories have become indispensable in modern approaches and applications of homological algebra. We just wish to mention one further feature that we will possibly touch upon at the end of this chapter: it can happen that a right exact functor that is not an equivalence induces an equivalence between the derived categories. Two abelian categories whose derived categories are equivalent are called derived equivalent. The question when two abelian categories are derived equivalent leads to a surprising and fascinating theory, called tilting theory.

3.1 The homotopy category $\mathrm{K}(\mathcal{A})$

Before introducing the derived category, which considers chain complexes up to quasi-isomorphism, we construct a category mediating between $\mathrm{Ch}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$. This is the *homotopy category* $\mathrm{K}(\mathcal{A})$, consisting roughly of chain complexes considered up to chain homotopy equivalence.

Lemma 3.1.1. *Let $\mathcal{A}$ be an abelian category and $C_*, D_* \in \mathrm{Ch}(\mathcal{A})$. Then the collection of null-homotopic chain maps forms a subgroup*

$$\mathrm{Hom}^{\mathrm{null}}_{\mathrm{Ch}(\mathcal{A})}(C_*, \mathcal{D}_*) \subset \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, \mathcal{D}_*)\,. \tag{32}$$

Proof. Let $f_*, g_* \colon C_* \to D_*$ be two chain maps with null homotopies H_1, H_2, meaning that $dH_1 + H_1 d = f_*$ and $dH_2 + H_2 d = g_*$. Then $f_* - g_*$ is null homotopic with null homotopy $H_1 - H_2$:

$$d(H_1 - H_2) + (H_1 - H_2)d = dH_1 + H_1 d - dH_2 - H_2 d = f_* - g_* \,.$$

Thus $\mathrm{Hom}^{\mathrm{null}}_{\mathrm{Ch}(\mathcal{A})}(C_*, \mathcal{D}_*) \subset \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, \mathcal{D}_*)$ is closed under addition and inverses, and therefore an abelian subgroup. □

Definition 3.1.2. Let $\mathcal{A}$ be an abelian category and $C_*, D_* \in \mathrm{Ch}(\mathcal{A})$. We define the abelian group

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C_*, D_*) \coloneqq \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, \mathcal{D}_*)/\,\mathrm{Hom}^{\mathrm{null}}_{\mathrm{Ch}(\mathcal{A})}(C_*, \mathcal{D}_*) \tag{33}$$

as the quotient abelian group of the inclusion (32).

Lemma 3.1.3. *Let $\mathcal{A}$ be an abelian category.*

(1) Let $g_* \colon B_* \to C_*$ be a chain map in $\mathcal{A}$ with null homotopy H. Let $f_* \colon A_* \to B_*$ and $h_* \colon C_* \to D_*$ be chain maps. Then the chain map $h_* \circ g_* \circ f_* \colon A_* \to D_*$ is null homotopic with null homotopy $h_* \circ H \circ f_*$.

(2) The composition of chain maps $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(B_*, C_*) \times \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(\mathcal{A})(C_*, D_*) \to \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(B_*, D_*)$ *induces a bilinear map*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(B_*, C_*) \times \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C_*, D_*) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(B_*, D_*)\,. \tag{34}$$

Proof. For part (1), we note that $d \circ (h_* \circ H \circ f_*) = h_* \circ dH \circ f_*$ since h_* is a chain map, and similarly $(h_* \circ H \circ f_*) \circ d = h_* \circ Hd \circ f_*$. Thus

$$d \circ (h_* \circ H \circ f_*) + (h_* \circ H \circ f_*) \circ d = h_* \circ (dH + Hd) \circ f_* = h_* \circ g_* \circ f_* \,,$$

as desired.

For part (2), we note that if either $f_* \colon B_* \to C_*$ or $g_* \colon C_* \to D_*$ is null homotopic, then so is $g_* \circ f_*$ by part (1). Composition in $\mathrm{Ch}(\mathcal{A})$ thus induces the map (34), and one readily checks that it is bilinear. □

Definition 3.1.4. Let $\mathcal{A}$ be an abelian category. The homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ is defined as the category with

- objects given by chain complexes in $\mathcal{A}$,
- given two chain complexes $C_*, D_* \in K(\mathcal{A})$, the set of morphisms in $K(\mathcal{A})$ given by the set (33),
- and the composition map given by the map (34).

We note that the associativity of the composition in $\mathrm{K}(\mathcal{A})$ is induced by the associativity of the composition in $\mathrm{Ch}(\mathcal{A})$.

Lemma 3.1.5. *Let $\mathcal{A}$ be an abelian category. The homotopy category* $\mathrm{K}(\mathcal{A})$ *is additive and the apparent functor* $\mathrm{Ch}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ *is an additive functor.*

Proof. It follows from Definition 3.1.2 and Lemma 3.1.3 that $\mathrm{K}(\mathcal{A})$ is Ab-enriched. We next show that $\mathrm{K}(\mathcal{A})$ admits coproducts. Let $A_*, C_*, D_* \in \mathrm{K}(\mathcal{A})$ and denote by $C_* \oplus D_*$ the direct sum in $\mathrm{Ch}(\mathcal{A})$. The inclusion morphisms $C_*, D_* \to C_* \oplus D_*$ induce a diagram of abelian groups

$$\begin{tikzcd}
\mathrm{Hom}^{\mathrm{null}}_{\mathrm{Ch}(\mathcal{A})}(C_* \oplus D_*, A_*) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}^{\mathrm{null}}_{\mathrm{Ch}(\mathcal{A})}(C_*, A_*) \times \mathrm{Hom}^{\mathrm{null}}_{\mathrm{Ch}(\mathcal{A})}(D_*, A_*) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_* \oplus D_*, A_*) \arrow[r, "\simeq"] \arrow[d, two heads] & \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, A_*) \times \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(D_*, A_*) \arrow[d, two heads] \\
\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C_* \oplus D_*, A_*) \arrow[r] & \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C_*, A_*) \times \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(D_*, A_*)
\end{tikzcd}$$

where the middle horizontal morphism is an equivalence by the universal property of the coproduct in $\mathrm{Ch}(\mathcal{A})$. That the top horizontal morphism is an equivalence follows from the observation that a chain map $(f_*, g_*) \colon C_* \oplus D_* \to A_*$ is null homotopic if and only if f_* and g_* are both null homotopic. By the 5-Lemma, or alternatively by the universal properties of the pushouts, it follows that the bottom morphism is an isomorphism. The unique existence of morphisms in the definition of coproduct amounts exactly to the bottom horizontal morphism in the above diagram being a bijection. We have thus shown that $C_* \oplus D_*$ is a coproduct in $\mathrm{K}(\mathcal{A})$.

The functor $\mathrm{Ch}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ is given by mapping each chain complex to itself and on Hom sets by the surjection $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C_*, D_*)$, which is a group homomorphism. Thus the functor is by definition additive. $\square$

Lemma 3.1.6. *The functor* $\mathrm{Ch}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ *maps chain homotopy equivalences to isomorphisms in* $\mathrm{K}(\mathcal{A})$.

Proof. If f_* is a chain homotopy equivalence, then there exists $g_* \colon D_* \to C_*$, such that $g_* \circ f_*$ is chain homotopic to id_{C_*} and $f_* \circ g_*$ is chain homotopic to id_{D_*}. Thus $g_* \circ f_* = \mathrm{id}_{C_*}$ and $f_* \circ g_* = \mathrm{id}_{D_*}$ in $\mathrm{K}(\mathcal{A})$. $\square$

Theorem 3.1.7 (The universal property of $\mathrm{K}(\mathcal{A})$)**.** *Let* $F \colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{C}$ *be a functor that maps chain homotopy equivalences to isomorphisms. Then F factors uniquely through* $\mathrm{Ch}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$*:*

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathcal{C} \\
\mathrm{K}(\mathcal{A}) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Proof. We show in Lemma 3.1.8 that if F maps chain homotopy equivalences to isomorphisms, then $F(f_*) = 0$ for any null homotopic chain map f_*. It then follows that $F \colon \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*) \to \mathrm{Hom}_{\mathcal{C}}(F(C_*), F(D_*))$ factors uniquely through the quotient $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*) \twoheadrightarrow \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C_*, D_*)$, showing the assertion. $\square$

Recall the construction of the mapping cylinder $\mathrm{cyl}(f_*)$ of a chain map $f_*\colon C_* \to D_*$: We have $\mathrm{cyl}(f_*)_n = D_n \oplus C_{n-1} \oplus C_n$ and

$$d_n = \begin{pmatrix} d_n^D & -f_{n-1} & 0 \\ 0 & -d_{n-1}^C & 0 \\ 0 & \mathrm{id}_{C_{n-1}} & d_n^C \end{pmatrix} .$$

As shown on Sheet 6, exercise 4, the apparent inclusion

$$\alpha_*\colon D_* \to \mathrm{cyl}(f_*), \quad x \mapsto (x, 0, 0)$$

is a chain homotopy equivalence. The chain homotopy inverse of α_* is given by

$$\beta_*\colon \mathrm{cyl}(f_*) \to D_*, \quad (x, y, z) \mapsto x + f_n(z) .$$

There is a further chain map

$$\alpha'_*\colon C_* \to \mathrm{cyl}(f_*), z \mapsto (0, 0, z)$$

and we have $\beta_* \circ \alpha'_* = f_*$.

Lemma 3.1.8. *Let $F\colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{C}$ be a functor that maps chain homotopy equivalences to isomorphisms. Let $f_*, g_*\colon C_* \to D_*$ be two chain homotopic chain map. Then $F(f_*) = F(g_*)$.*

Proof. We first consider the mapping cylinder $\mathrm{cyl}(\mathrm{id}_{\mathcal{C}_*})$. We denote α_*, α'_* and β_* as above. We have $\alpha_* \circ \beta_* = \mathrm{id}_{C_*}$. Since α_*, β_* are chain homotopy equivalences, by assumption, $F(\alpha_*), F(\beta_*)$ are isomorphisms. From

$$F(\alpha_*) \circ F(\beta_*) = F(\alpha_* \circ \beta_*) = F(\mathrm{id}_{C_*}) = \mathrm{id}_{F(C_*)}$$

it follows that $F(\beta_*) = F(\alpha_*)^{-1}$. We thus obtain

$$F(\alpha'_*) = F(\alpha_*) \circ F(\beta_*) \circ F(\alpha'_*) = F(\alpha_*) \circ F(\beta_* \circ \alpha'_*) = F(\alpha_*) \circ F(\mathrm{id}_{C_*}) = F(\alpha_*) .$$

Let H be a chain homotopy from f_* to g_*. There is a chain map

$$\gamma_* = (f_*, H, g_*)\colon \mathrm{cyl}(\mathrm{id}_{C_*}) \to D_*$$

with $\gamma_* \circ \alpha_* = f_*$ and $\gamma_* \circ \alpha'_* = g_*$. We thus have

$$F(g_*) = F(\gamma_* \circ \alpha'_*) = F(\gamma_*) \circ F(\alpha'_*) = F(\gamma_*) \circ F(\alpha_*) = F(\gamma_* \circ \alpha_*) = F(f_*) ,$$

as desired. □

Lemma 3.1.9. *Let $i \in \mathbb{Z}$. The homology functor $H_i(\text{-})\colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ factors through $\mathrm{K}(\mathcal{A})$.*

Proof. The homology functor $H_i(\text{-})$ maps chain homotopy equivalences to isomorphisms by Lemma 1.5.19. The assertion thus follows from Theorem 3.1.7. □

3.1.1 Cocones and chain homotopy

We discuss some results concerning the interaction between the cone construction of a chain map and chain homotopy. A main result will be an exact sequence of Hom groups in the homotopy category, see Lemma 3.1.14.

Recall that the cone of a chain map $f_*\colon C_* \to D_*$ is given by the chain complex $\mathrm{cone}(f_*)$ with

- n-chains $\mathrm{cone}(f_*)_n = D_n \oplus C_{n-1}$
- and differential

$$d = \begin{pmatrix} d_n^D & -f_{n-1} \\ 0 & -d_{n-1}^C \end{pmatrix} .$$

There are canonical chain maps

$$D_* \to \mathrm{cone}(f_*), d \mapsto (d, 0)$$

and

$$\mathrm{cone}(f_*) \to C_*[1], (d, c) \mapsto c \,,$$

where $C_*[1]$ is the shifted chain complex with n-chains $(C_*[1])_n = C_{n-1}$ and differential $d^{C[1]} = -d^C$.

Definition 3.1.10. The cocone of a chain map $\mathrm{cocone}(f_*)$ is defined as the shifted cone

$$\mathrm{cocone}(f_*) = \mathrm{cone}(f_*)[-1] \,,$$

meaning that its n-chains are $\mathrm{cocone}(f_*)_n = D_{n+1} \oplus C_n$ and the differential is

$$d = \begin{pmatrix} -d_{n+1}^D & f_n \\ 0 & d_n^C \end{pmatrix} .$$

Note that also $\mathrm{cocone}(f_*)[1] = \mathrm{cone}(f_*)$ and that there are canonical chain maps

$$D_*[-1] \to \mathrm{cocone}(f_*) \to C_* \,.$$

Lemma 3.1.11. *Suppose that f_* is chain homotopic to g_*. Then there exist isomorphisms in* $\mathrm{Ch}(\mathcal{A})$

$$\mathrm{cone}(f_*) \simeq \mathrm{cone}(g_*)$$

and

$$\mathrm{cocone}(f_*) \simeq \mathrm{cocone}(g_*) \,.$$

Proof. We only prove the isomorphisms between the cones. The isomorphisms between the cocones follows from the observation that $[1]\colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ is an isomorphism of categories and thus preserves and reflects isomorphisms.

Let $f_*, g_*\colon C_* \to D_*$ be chain maps and $H = (H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ with $dH + Hd = f_* - g_*$ the chain homotopy from f_* to g_*.

We define $h_*\colon \operatorname{cone}(f_*) \to \operatorname{cone}(g_*)$ by

$$h_n = \begin{pmatrix} \mathrm{id}_{D_n} & H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} .$$

and $\tilde{h}_*\colon \operatorname{cone}(g_*) \to \operatorname{cone}(f_*)$ by

$$\tilde{h}_n = \begin{pmatrix} \mathrm{id}_{D_n} & -H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} .$$

Then h_* is a chain map:

$$\begin{aligned} dh_n &= \begin{pmatrix} d^D_n & -f_{n-1} \\ 0 & -d^C_{n-1} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{D_n} & H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} \\ &= \begin{pmatrix} d^D_n & d^D_n H_{n-1} - f_{n-1} \\ 0 & -d^C_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} d^D_n & -H_{n-2} d^C_n - g_{n-1} \\ 0 & -d^C_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_{D_{n-1}} & H_{n-2} \\ 0 & \mathrm{id}_{C_{n-2}} \end{pmatrix} \begin{pmatrix} d^D_n & -g_{n-1} \\ 0 & -d^C_{n-1} \end{pmatrix} \\ &= h_{n-1} d \end{aligned}$$

In the third equality above, we used that $d^D_n H_{n-1} + H_{n-2} d^C_n = f_{n-1} - g_{n-1}$ and so $d^D_n H_{n-1} - f_{n-1} = -H_{n-2} d^C_n - g_{n-1}$.

A similar computation shows that $\tilde{h}_*$ is a chain map.

The chain maps h_* and $\tilde{h}_*$ are inverse: We have

$$\tilde{h}_n \circ h_n = \begin{pmatrix} \mathrm{id}_{D_n} & H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{D_n} & -H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{D_n} & H_{n-1} - H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} = \mathrm{id}_{\operatorname{cone}(f_*)_n} .$$

Similarly, we have

$$h_n \circ \tilde{h}_n = \begin{pmatrix} \mathrm{id}_{D_n} & -H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{D_n} & H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{D_n} & -H_{n-1} + H_{n-1} \\ 0 & \mathrm{id}_{C_{n-1}} \end{pmatrix} = \mathrm{id}_{\operatorname{cone}(g_*)_n} .$$

□

Remark 3.1.12. Lemma 3.1.11 implies that the cone or cocone of a morphism in the homotopy category $\mathrm{K}(\mathcal{A})$ is well defined up to isomorphism.

Lemma 3.1.13. *Consider a chain map $f_*\colon C_* \to D_*$ and the induced chain map $g_*\colon \operatorname{cocone}(f_*) \to C_*$.*

(1) The chain map $f_* \circ g_*$ is null homotopic.

(2) Let $a_*\colon B_* \to C_*$ and H a null homotopy of $f_* \circ a_*$. Then there is an induced chain map $b_*\colon B_* \to \mathrm{cocone}(f_*)$ such that the following diagram commutes in $\mathrm{Ch}(\mathcal{A})$:

$$\begin{tikzcd} B_* \arrow[dr, "a_*"] \arrow[dd, "b_*"'] & & \\ & C_* \arrow[r, "f_*"] & D_* \\ \mathrm{cocone}(f_*) \arrow[ur, "g_*"'] & & \end{tikzcd}$$

Proof. For part (1), we note that the null homotopy is given by $H_n = (0, \mathrm{id}_{D_n})\colon \mathrm{cone}(f_*)_n = C_{n-1} \oplus D_n \to D_n$.

For part (2), we define b_* via $b_n = (a_n, H_n)\colon B_n \to D_{n+1} \oplus C_n = \mathrm{cocone}(f_*)$. We have

$$d \circ b_* = \begin{pmatrix} -d^D_{n+1} & f_n \\ 0 & d^C_n \end{pmatrix} \begin{pmatrix} H_n \\ a_n \end{pmatrix} = (-d^D_{n+1} H_n + f_n a_n, d^C_n a_n) = (H_{n-1} d^B_n, a_{n-1} d^B_n) = b_* \circ d\,,$$

showing that b_* is a chain map. The commutativity of the diagram is immediate from the definitions of g_* and h_*. □

Lemma 3.1.14. *Let $f_*\colon C_* \to D_*$ and $g_*\colon \mathrm{cocone}(f_*) \to C_*$ be the induced map. For any chain complex $B_* \in \mathrm{K}(\mathcal{A})$ the sequence*

$$\mathrm{Hom}_{K(\mathcal{A})}(B_*, \mathrm{cocone}(f_*)) \xrightarrow{g_* \circ (\text{-})} \mathrm{Hom}_{K(\mathcal{A})}(B_*, C_*) \xrightarrow{f_* \circ (\text{-})} \mathrm{Hom}_{K(\mathcal{A})}(B_*, D_*)$$

of abelian groups is exact[4].

Proof. Firstly, the composition vanishes since $f_* \circ g_*$ is null homotopic. Given $a_*\colon B_* \to C_*$ such that $f_* \circ a_*$ is null homotopic, we can find by Lemma 3.1.13 a chain map $b_*\colon B_* \to \mathrm{cocone}(f_*)$ such that $g_* \circ b_* = a_*$. □

Lemma 3.1.15. *Let $f_*\colon C_* \to D_*$ be a chain map. There exists an isomorphism $e_*\colon C_* \simeq \hat{C}_*$ in $\mathrm{K}(\mathcal{A})$ and an epimorphism $p_*\colon \hat{C}_* \to D_*$ in $\mathrm{Ch}(\mathcal{A})$ such that $p_* \circ e_* = f_*$ in $\mathrm{Ch}(\mathcal{A})$, i.e. the following diagram commutes:*

$$\begin{tikzcd} C_* \arrow[rr, "e_*"] \arrow[dr, "f_*"'] & & \hat{C}_* \arrow[dl, two heads, "p_*"] \\ & D_* & \end{tikzcd}$$

Proof. Let $\hat{C}_* = \mathrm{cocone}(D_* \to \mathrm{cone}(f_*))$ (this is a dual version of the mapping cylinder) with $\hat{C}_n = D_n \oplus D_{n+1} \oplus C_n$. The chain homotopy equivalence e_* is given by the inclusion $C_* \to \hat{C}_*, z \mapsto (0, 0, z)$. The epimorphism $p_*\colon \hat{C}_* \to D_*$ is the induced map from the cocone, given explicitly by $(x, y, z) \mapsto x + f(z)$. □

[4]Note that the sequence is in general not short exact, i.e. cannot be extended with 0's to the left and right.

Example 3.1.16. Let D_* be a chain complex and consider the chain map $0\colon 0 \to D_*$. Then a replacement epimorphism of 0 is the projection from the cocone of id_{D_*}:

$$\begin{tikzcd} 0 \arrow[rr, "\simeq"] \arrow[dr] & & \operatorname{cocone}(\mathrm{id}_{D_*}) \arrow[dl, two heads] \\ & D_* & \end{tikzcd}$$

The top horizontal chain map is a chain homotopy equivalence.

3.2 Localization of categories

Definition 3.2.1. Let $\mathcal{C}$ be a category and W a collection of morphisms in $\mathcal{C}$. The localization of $\mathcal{C}$ at W is a category $\mathcal{C}[W^{-1}]$ together with a functor $\pi\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that

- π maps the morphisms in W to isomorphisms in $\mathcal{D}$ and
- for any category $\mathcal{E}$ and functor $F\colon \mathcal{C} \to \mathcal{E}$ mapping W to isomorphisms in $\mathcal{E}$ there exists a unique factorization through π:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathcal{E} \\ \mathcal{C}[W^{-1}] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Remark 3.2.2. If the localization of a category at a collection of morphisms exists, then the localization is unique up to isomorphism of categories. In general, localizations of categories do not have to exist in a given universe, see also Weibel for a discussion.

Example 3.2.3. Theorem 3.1.7 amounts to the assertion that the homotopy category $\mathrm{K}(\mathcal{A})$ is the localization of $\mathrm{Ch}(\mathcal{A})$ at the collection of chain homotopy equivalences H:

$$\mathrm{K}(\mathcal{A}) \simeq \mathrm{Ch}(\mathcal{A})[H^{-1}]\,.$$

3.2.1 The calculus of fractions

Example 3.2.4. An Ab-enriched category $\mathcal{C}$ with a single object $*$ amounts to a ring $R = \operatorname{Hom}_{\mathcal{C}}(*, *)$. We denote $\mathcal{C} = \mathcal{C}_R$

Suppose that R is commutative. Let $S \subset R$ be a multiplicatively closed subset containing 1_R. The localization of R at S is the ring $S^{-1}R$ whose elements are equivalence classes of fractions $s^{-1}r$ with $s \in S$ and $r \in R$, where

$$s_1^{-1}r_1 \sim s_2^{-1}r_2$$

if there exists $t \in S$ such that

$$t(s_1 r_2 - s_2 r_1) = 0\,.$$

If R is an integral domain, then the above relation simplifies to $s_1 r_2 = s_2 r_1$.

The category corresponding to $S^{-1}R$ is a localization of $\mathcal{C}_R$:

$$\mathcal{C}_{S^{-1}R} \simeq \mathcal{C}_R[S^{-1}]\,.$$

To see this, consider a category $\mathcal{D}$. A functor $F\colon \mathcal{C}_R \to \mathcal{D}$ amounts to a choice of object $d \in \mathcal{D}$ and a monoid morphism $R \to M := \mathrm{Hom}_{\mathcal{D}}(d,d)$. If F maps S to isomorphisms in $\mathcal{D}$, meaning invertible elements in the monoid M, we can define the functor

$$\mathcal{C}_{S^{-1}R} \to \mathcal{D}, s^{-1}r \mapsto F(s)^{-1}F(r) \in M\,.$$

This functor is well defined: If $t(s_1r_2 - s_2r_1) = 0$, then $F(t)F(s_1)F(r_2) = F(t)F(s_2)F(r_1)$. Since $F(t)$ is an isomorphism, it follows that $F(s_1)F(r_2) = F(s_2)F(r_1)$. We thus also have $F(s_1)^{-1}F(r_1) = F(s_2)^{-1}F(r_2)$ in M.

To see uniqueness, we note that $F(s^{-1})F(s) = F(s^{-1}s) = F(1_R) = \mathrm{id}_d$, so that F must satisfy $F(s^{-1}) = F(s)^{-1}$. Hence, F must be given by the above assignment.

The above is a proto-typical example for the *calculus of fractions* for describing localizations. To be able to describe a localization in terms of such a calculus of fractions, we will need in general additional conditions, given as follows:

Definition 3.2.5. Let $\mathcal{C}$ be a category. A multiplicative systems S in $\mathcal{C}$ consists of a collection of morphisms sastisfying the following:

(1) S is closed under composition and contains id_x for all $x \in \mathcal{C}$.

(2) (Ore condition) Let $g\colon x \to y$ and $t\colon z \to y$ be morphisms in $\mathcal{C}$ such that t lies in S. Then there exists an object $w \in \mathcal{C}$ and morphisms $s\colon w \to x$ and $f\colon w \to z$ with $s \in S$, such that the following square commutes:

$$\begin{array}{ccc} w & \xrightarrow{f} & z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ x & \xrightarrow{g} & y \end{array}$$

The dual condition, where the positions of t, s and g, f are swapped also holds.

(3) Let $f, g\colon x \to y$ be morphisms in $\mathcal{C}$. Then there exists $s \in S$ with $s \circ f = s \circ g$ if and only if there exists $t \in S$ with $f \circ t = g \circ t$.

Remark 3.2.6. If R is a commutative ring and $S \subset R$ a multiplicatively closed subset containing 1_R, it defines a multiplicative system in the corresponding one object category $\mathcal{C}_R$: the ore condition follows from $tg = gt$ by the commutativity of R, and the cancellation property (3) follows from $fs = sf = sg = gs$. If R is non-commutative, the conditions are however not automatic. Given these, we can define the localization $S^{-1}R$ as the ring consisting of fractions rs^{-1}, subject to the equivalence relation $r_1s_1^{-1} \sim r_2s_2^{-1}$ if $tr_1s_2 = tr_2s_1$ for some $t \in S$. The multiplication is defined by

$$r_1s_1^{-1} \cdot r_2s_2^{-1} = r_1rt^{-1}s_2^{-1}\,,$$

where we use the Ore condition to find $t \in S$ and $r \in R$ with "$s_1^{-1}r_2 = rt^{-1}$", meaning $r_2t = s_1r$. We show below that this composition is well defined. Similarly to Example 3.2.4, one finds that $\mathcal{C}_{S^{-1}R} \simeq \mathcal{C}_R[S^{-1}]$ describes the localization of $\mathcal{C}_R$.

Construction 3.2.7 (Gabriel-Zisman localization)**.** Let $\mathcal{C}$ be a category with a multiplicative system S. We describe a category $S^{-1}\mathcal{C}$:

The objects of $S^{-1}\mathcal{C}$ are simply the objects of $\mathcal{C}$. Given $x, y \in \mathcal{C}$, we specify $\mathrm{Hom}_{S^{-1}\mathcal{C}}(x, y)$ to be the collection of all diagrams

$$fs^{-1} := \left(x \xleftarrow{s} z \xrightarrow{f} y\right)$$

in $\mathcal{C}$ with $s \in S$, modulo the equivalence relation

$$\left(x \xleftarrow{s} z \xrightarrow{f} y\right) \quad \sim \quad \left(x \xleftarrow{s'} z' \xrightarrow{f'} y\right)$$

if there exists a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
& z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
x & w \arrow[l, "t"'] \arrow[u] \arrow[r] \arrow[d] & y \\
& z' \arrow[ul, "s'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

with $t \in S$.

The composite of fs^{-1} and $f'(s')^{-1}$ is defined as $fgt^{-1}(s')^{-1} = (fg)(s't)^{-1}$, where we used the Ore condition to find $t \in S$ and g with $f't = sg$. We can illustrate composition diagrammatically as follows:

$$\begin{tikzcd}
& & u \arrow[dl, "t"] \arrow[dr, "g"] & & \\
& v \arrow[dl, "s"] \arrow[dr, "f"] & & w \arrow[dl, "s'"] \arrow[dr, "f'"] & \\
x & & y & & z
\end{tikzcd}$$

To check that composition is **well-defined**, we need to verify two things:

Firstly, if we choose a different square via the ore condition (with top denotes u'), we can get the same composition. This can be seen as follows. Applying the ore condition to the maps $u \to v \leftarrow u'$, we get a diagram

$$\begin{tikzcd}
& o \arrow[dl, "\tilde{t}"'] \arrow[dr, "\tilde{g}"] & \\
u \arrow[d, "t"'] \arrow[drr, "g" near end] & & u' \arrow[d, "g'"] \arrow[dll, "t'" near end] \\
v \arrow[dr, "f"] & & w \arrow[dl, "s'"] \\
& y &
\end{tikzcd}$$

with $\tilde{t} \in S$. We have

$$s'g'\tilde{g} = ft'\tilde{g} = ft\tilde{t} = s'g\tilde{t}\,.$$

By the third condition in Definition 3.2.5, we can find $a\colon o' \to o$ in S with $g'\tilde{g}a = g\tilde{t}a$. We thus find that the diagram

$$\begin{tikzcd}
& & o' \arrow[dl, "\tilde{t}a"'] \arrow[dr, "\tilde{g}a"] & & \\
& u \arrow[d, "t"'] \arrow[drr, "g" near end] & & u' \arrow[d, "g'"] \arrow[dll, "t'" near end] & \\
& v \arrow[dl, "s"'] \arrow[dr, "f"] & & w \arrow[dl, "s'"] \arrow[dr, "f'"] & \\
x & & y & & y
\end{tikzcd}$$

commutes. This exhibits the desired equivalence between the fractions $f'g(st)^{-1}$ and $f'g'(st')^{-1}$.

Secondly, if we replace fs^{-1} by an equivalent fraction, the resulting fraction after composition with $f'(s')^{-1}$ is also equivalent. For this consider a diagram

$$\begin{tikzcd}
& v \arrow[dl, "s"'] \arrow[dr, "f"] & \\
x & \tilde{v} \arrow[l, "\tilde{f}"'] \arrow[r, "\tilde{f}"] \arrow[u] \arrow[d] & y \\
& v' \arrow[ul, "\bar{s}"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

Th statement to prove is that $f'(s')^{-1} \circ \bar{f}\bar{s}^{-1} = f'(s')^{-1} \circ fs^{-1}$, but we only show 'half' of this, namely that $f'(s')^{-1} \circ \tilde{f}\tilde{s}^{-1} = f'(s')^{-1} \circ fs^{-1}$. The missing half follows from a similar argument. Consider the following diagram, where the top square was chosen via the Ore condition, and hence satisfies that $\tilde{t} \in S$.

$$\begin{tikzcd}
 & m \arrow[dl, "\tilde{t}"'] \arrow[dr] & & & \\
\tilde{v} \arrow[dr] \arrow[dd, "\tilde{s}"'] & & u \arrow[dl, "t"'] \arrow[dr, "g"] & & \\
 & v \arrow[dl, "s"'] \arrow[dr, "f"] & & w \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\
x & & y & & z
\end{tikzcd}$$

This shows the equality $f'(s')^{-1} \circ \tilde{f}\tilde{s}^{-1} = f'(s')^{-1} \circ fs^{-1}$.

Finally, we remark that the composition in $S^{-1}\mathcal{C}$ is associative and the units are given by the fractions $1_x 1_x^{-1} \colon x \to x$.

The following theorem states that localizations by multiplicative systems exist and are computed as the Gabriel-Zisman localization. In this case, the localization is also said to admit a *calculus of fractions*.

Theorem 3.2.8. *Let $\mathcal{C}$ be a category and S a multiplicative system. Then there is a canonical functor $\mathcal{C} \to S^{-1}\mathcal{C}$ exhibiting $S^{-1}\mathcal{C}$ as the localization $\mathcal{C}[S^{-1}]$.*

Proof. We define the functor $\mathcal{C} \to S^{-1}\mathcal{C}$ as the identity on objects and mapping each morphism $f \colon x \to y$ to the faction $f\,\mathrm{id}_x^{-1}$. If $f \in S$, then the diagram

$$\begin{tikzcd}
 & & x \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_x"] & & \\
 & x \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "f"] & & x \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}_x"] & \\
x & & y & & x
\end{tikzcd}$$

shows that $(\mathrm{id}_y \circ f^{-1}) \circ (f\,\mathrm{id}_x^{-1}) = \mathrm{id}_x\,\mathrm{id}_x^{-1}$. Thus $(f\,\mathrm{id}_x^{-1})$ is an isomorphism in $S^{-1}\mathcal{C}$.

Let $F \colon \mathcal{C} \to \mathcal{E}$ be a functor mapping S to isomorphisms in $\mathcal{E}$. Then F uniquely factors through $S^{-1}\mathcal{C}$ via the functor $\tilde{F} \colon S^{-1}\mathcal{C} \to \mathcal{E}$, given by F on objects and mapping fs^{-1} to $F(f)F(s)^{-1}$. We show that $\tilde{F}$ is well defined: applying F to a commutative diagram

$$\begin{tikzcd}
 & z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
x & w \arrow[l, "t"'] \arrow[r] \arrow[u] \arrow[d] & y \\
 & z' \arrow[ul, "s'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

and using the invertibility of $F(s), F(t), F(s')$, one finds that $F(f)F(s)^{-1} = F(f')F(s')^{-1}$.

$\tilde{F}$ is further a functor, meaning it commutes with composition: given a diagram

$$\begin{tikzcd} & & u \arrow[dl, "t"'] \arrow[dr, "g"] & & \\ & v \arrow[dl, "s"'] \arrow[dr, "f"] & & w \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ x & & y & & z \end{tikzcd}$$

we have $F(f)F(t) = F(s')F(g)$ and thus $F(s')^{-1}F(f) = F(g)F(t)^{-1}$, and therefore

$$F(f')F(s')^{-1}F(f)F(s)^{-1} = F(f')F(g)F(t)^{-1}F(s)^{-1} = F(f'g)F(st)^{-1}\,.$$

□

3.2.2 Quasi-isomorphisms as a multiplicative system

The homology functors $H_i(\text{-})$ descend to functors $H_i\colon\ \mathrm{K}(\mathcal{A}) \to \mathcal{A}$, see Lemma 3.1.9.

Definition 3.2.9. A morphism $[f_*]$ in $\mathrm{K}(\mathcal{A})$ is called a quasi-isomorphism if $H_i([f_*])$ is an isomorphism for all $i \in \mathbb{Z}$. We denote by Q the collection of quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$.

Theorem 3.2.10. *The collection Q of quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$ defines a multiplicative system. In particular, the localization $\mathrm{K}(\mathcal{A})[Q^{-1}]$ exists and is described by the calculus of fractions in the homotopy category.*

Definition 3.2.11. Let $\mathcal{A}$ be an abelian category. The derived category of $\mathcal{A}$ is defined as the localization

$$\mathcal{D}(\mathcal{A}) \coloneqq \mathrm{K}(\mathcal{A})[Q^{-1}]\,.$$

The proof of Theorem 3.2.10 makes heave use of the results from Section 3.1.1.

Remark 3.2.12. The collection of quasi-isomorphisms does not define a multiplicative system on the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes if $\mathcal{A} \neq 0$. Let $X \in \mathcal{A}$ be a non-zero object, and consider X as a chain complex concentrated in degree 0. Then the Ore condition fails for the diagram

$$\begin{tikzcd} & X \arrow[d, hook] \\ 0 \arrow[r] & \mathrm{cone}(\mathrm{id}_X) \end{tikzcd}$$

where the horizontal morphisms is a quasi-isomorphism. This is because the right vertical morphism is a monomorphism and thus any morphism $C_* \to X$ whose composite $C_* \to \mathrm{cone}(\mathrm{id}_X)$ vanishes must already vanish, and therefore cannot be a quasi-isomorphism.

Since $0 \to \mathrm{cone}(\mathrm{id}_X)$ is a chain homotopy equivalence, this example equally shows that the collection of chain homotopy equivalences also does not define a multiplicative system on $\mathrm{Ch}(\mathcal{A})$.

Proof of Theorem 3.2.10. The fact that $H_i(\text{-})$ is a functor implies that quasi-isomorphisms are closed under composition.

We next show that the Ore condition is satisfied. Consider a diagram in $\mathrm{K}(\mathcal{A})$

$$\begin{tikzcd} & D_* \arrow[d, "t_*"] \\ B_* \arrow[r, "g_*"] & C_* \end{tikzcd}$$

where t_* is a quasi-isomorphism. By Lemma 3.1.15, we may assume that g_* is an epimorphism in $\mathrm{Ch}(\mathcal{A})$. Let $A_* = B_* \times_{C_*} D_*$ be the pullback in $\mathrm{Ch}(\mathcal{A})$. Then the sequence

$$A_* \longrightarrow B_* \oplus D_* \xrightarrow{(-g_*, t_*)} C_*$$

in $\mathrm{Ch}(\mathcal{A})$ is short exact by the folding Lemma 1.4.10 and the fact that $\mathrm{Ch}(\mathcal{A})$ is abelian (kernels along epis give rise to SES). Consider the associated LES:

$$\cdots \to H_n(A_*) \to H_n(B_*) \oplus H_n(D_*) \xrightarrow{(-H_n(g_*), H_n(t_*))} H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \to \dots$$

Since $H_n(t_*)$ is an isomorphism the map $(-H_n(g_*), H_n(t_*))$ must be an epimorphism in $\mathcal{A}$. Thus its image is $H_n(C_*)$ and therefore the connecting homomorphism ∂ must vanish by exactness. We thus find that for any $n \in \mathbb{Z}$ there is a split SES

$$0 \to H_n(A_*) \to H_n(B_*) \oplus H_n(D_*) \xrightarrow{(-H_n(g_*), H_n(t_*))} H_n(C_*) \to 0$$

with section $(0, H_n(t_*)^{-1})$. Thus $H_n(A_*) \to H_n(B_*)$ is an isomorphism, meaning that $A_* \to B_*$ is a quasi-isomorphism. The commutative square

$$\begin{tikzcd} A_* \arrow[r] \arrow[d] & D_* \arrow[d, "t_*"] \\ B_* \arrow[r, "g_*"] & C_* \end{tikzcd}$$

thus shows that the first half of the Ore condition is satisfied. A similar argument, replacing epimorphisms with monomorphisms, shows that the second half of the Ore condition is also satisfied.

Let us show the cancellation property (3). Let $f_*, g_* \colon B_* \to C_*$ and $s_* \colon C_* \to D_*$ be a quasi-isomorphism with $s_* \circ f_* = s_* \circ g_*$. Consider the arising morphism $u_* \colon \operatorname{cocone}(s_*) \to C_*$. The fact that s_* is a quasi-isomorphism implies that $\operatorname{cocone}(s_*)$ has vanishing homology. Consider the exact sequence from Lemma 3.1.14:

$$\operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(B_*, \operatorname{cocone}(s_*)) \xrightarrow{u_* \circ (\text{-})} \operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(B_*, C_*) \xrightarrow{s_* \circ (\text{-})} \operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(B_*, D_*)\,.$$

Since $s_* \circ (f_* - g_*) = 0$, we can find $a_* \colon B_* \to \operatorname{cocone}(s_*)$ with $u_* \circ a_* = f_* - g_*$. Let $b_* \colon \operatorname{cocone}(a_*) \to B_*$ be the induced map. Then $a_* \circ b_* = 0$ in $\mathrm{K}(\mathcal{A})$, see Lemma 3.1.13.(1), and thus

$$(f_* - g_*) \circ b_* = u_* \circ a_* \circ b_* = 0\,,$$

as desired. The other direction in the cancellation property follows from a similar argument. $\square$

Lemma 3.2.13. *Consider two morphisms $f_1s_1^{-1}, f_2s_2^{-1}\colon x \to y$ in $\mathcal{D}(\mathcal{A})$. Then we can find equivalent fractions with a common denominator $s \in Q$.*

Proof. Using the Ore condition, we can find a commutative square

$$\begin{tikzcd} z \arrow[r, "t_1"] \arrow[d, "t_2"] & z_2 \arrow[d, "s_2"] \\ z_1 \arrow[r, "s_1"] & x \end{tikzcd}$$

with $t_2 \in Q$. We set $s = s_1t_2 \in Q$. The commutative diagram

$$\begin{tikzcd} & z_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ x & z \arrow[l, "s"] \arrow[u, "t_2"] \arrow[r, "f_1t_1"] \arrow[d, "\mathrm{id}_z"'] & y \\ & z \arrow[ul, "s"] \arrow[ur, "f_1t_1"'] & \end{tikzcd}$$

shows that $(f_1t_1)s^{-1} \sim f_1s_1^{-1}$. A similar argument shows that $(f_2t_2)s^{-1} \sim f_2s_2^{-1}$. $\square$

Proposition 3.2.14. *Let $\mathcal{A}$ be an abelian category. Then $\mathcal{D}(\mathcal{A})$ is an additive category and the localization functor $\mathrm{K}(\mathcal{A}) \to \mathcal{D}(A)$ is additive.*

Proof sketch. We first show that $\mathcal{D}(\mathcal{A})$ is Ab-enriched. Let $fs^{-1}, f't^{-1}\colon x \to y$ be two fractions. By Lemma 3.2.13, replacing $f't^{-1}$ by an equivalent fraction, we may assume that $s = t$. We define $fs^{-1} + f's^{-1} = (f + f')s^{-1}$. We leave to the reader to show that this is a well-defined addition on $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(x, y)$.

With this addition, composition is bilinear. There are two cases to consider: Firstly, let $fs^{-1}, f's^{-1}\colon y \to z$ and $gt^{-1}\colon x \to y$. Choosing an Ore square

$$\begin{tikzcd} & & u \arrow[dl, "s'"] \arrow[dr, "g'"] & & \\ & v \arrow[dl, "t"] \arrow[dr, "g"] & & w \arrow[dl, "s"] \arrow[dr, "{f,f'}"] & \\ x & & y & & z \end{tikzcd}$$

we see that

$$(f's^{-1}) \circ (gt^{-1}) + (fs^{-1}) \circ (gt^{-1}) = f'g'(ts')^{-1} + fg'(ts')^{-1} = (f' + f)g'(ts')^{-1}\,.$$

The second case considers $fs^{-1}, f's^{-1}\colon x \to y$ and $gt^{-1}\colon y \to z$. Composing gt^{-1} with fs^{-1} and with $f's^{-1}$ involves two Ore squares with possibly different denominators. We can find common denominators as follows. Applying the Ore condition, we find $a \in Q$ and h_1 with $f \circ a = t \circ h_1$:

$$\begin{tikzcd} & & u \arrow[dl, "a"] \arrow[dr, "h_1"] & & \\ & v \arrow[dl, "s"] \arrow[dr, "f"] & & w \arrow[dl, "t"] \arrow[dr, "g"] & \\ x & & y & & z \end{tikzcd}$$

Applying the Ore condition again to $v \xrightarrow{f' \circ a} y \xleftarrow{t} w$, we find $b \in Q$ and h_2 with $f' \circ a \circ b = t \circ h_2$, which we rearrange to an Ore square as follows:

$$\begin{tikzcd}
& & u' \arrow[dl, "ab"'] \arrow[dr, "h_2"] & & \\
& v \arrow[dl, "s"'] \arrow[dr, "f'"] & & w \arrow[dl, "t"'] \arrow[dr, "g"] & \\
x & & y & & z
\end{tikzcd}$$

We can replace the Ore square involving u with the Ore square

$$\begin{tikzcd}
u' \arrow[r, "h_1 b"] \arrow[d, "ab"'] & w \arrow[d, "t"] \\
v \arrow[r, "f"'] & y
\end{tikzcd}$$

Since composition does not depend on the choice of Ore square, the two compositions are given as follows:

$$(gt^{-1}) \circ (fs^{-1}) = (gh_1b)(sab)^{-1}\,,$$
$$(gt^{-1}) \circ (f's^{-1}) = (gh_2)(sab)^{-1}\,.$$

We thus have

$$(gt^{-1}) \circ (fs^{-1}) + (gt^{-1}) \circ (f's^{-1}) = (gh_1b + gh_2)\,(sab)^{-1} = (g(h_1b + h_2))\,(sab)^{-1}\,.$$

The following diagram commutes by bilinearity of the composition in $\mathrm{K}(\mathcal{A})$:

$$\begin{tikzcd}
& & u \arrow[dl, "ab"'] \arrow[dr, "h_1b+h_2"] & & \\
& v \arrow[dl, "s"'] \arrow[dr, "f+f'"] & & w \arrow[dl, "t"'] \arrow[dr, "g"] & \\
x & & y & & z
\end{tikzcd}$$

This shows that

$$(gt^{-1}) \circ ((f + f')s^{-1}) = g(h_1b + h_2)(sab)^{-1} = gt^{-1}) \circ (fs^{-1}) + (gt^{-1}) \circ (f's^{-1})\,,$$

as desired for bilinearity.

We leave the proof that $\mathcal{D}(\mathcal{A})$ admits products, given by the products in $\mathrm{Ch}(\mathcal{A})$, to the reader. Thus, $\mathcal{D}(A)$ is additive. Inspecting the definition of the Ab-enrichment of $\mathcal{D}(\mathcal{A})$, it is clear that the functor $\mathrm{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is additive. $\square$

3.2.3 Bounded derived categories

Definition 3.2.15. Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{K}^+(\mathcal{A}) \subset \mathrm{K}(\mathcal{A})$ the full subcategory consisting of bounded below chain complexes. We similarly denote by $\mathrm{K}^-(\mathcal{A}) \subset \mathrm{K}(\mathcal{A})$ the full subcategory consisting of bounded above chain complexes. We denote by $\mathrm{K}^b(\mathcal{A}) \subset \mathrm{K}(\mathcal{A})$ the full subcategory consisting of bounded (i.e. bounded above and below) chain complexes.

The proof of Theorem 3.2.10 shows with no changes that the collection of quasi-isomorphisms define mutliplicative systems on $\mathrm{K}^+(\mathcal{A}), \mathrm{K}^-(\mathcal{A}), \mathrm{K}^b(\mathcal{A})$.

Theorem 3.2.16. *The collections of quasi-isomorphisms Q define multiplicative systems on $\mathrm{K}^+(\mathcal{A}), \mathrm{K}^-(\mathcal{A}), \mathrm{K}^b(\mathcal{A})$. The corresponding localizations*

$$\mathcal{D}^+(\mathcal{A}) \coloneqq \mathrm{K}^+(\mathcal{A})[Q^{-1}], \quad \mathcal{D}^-(\mathcal{A}) \coloneqq \mathrm{K}^-(\mathcal{A})[Q^{-1}], \quad \mathcal{D}^b(\mathcal{A}) \coloneqq \mathrm{K}^b(\mathcal{A})[Q^{-1}]$$

thus exist and are described by the Gabriel-Zisman localizations.

Our next goal is to describe $\mathcal{D}^+(\mathcal{A})$ in the case that $\mathcal{A}$ admits enough projectives and $\mathcal{D}^-(\mathcal{A})$ in the case that $\mathcal{A}$ admits enough injectives.

Definition 3.2.17. Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Proj}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory consisting of projective objects and by $\mathrm{Inj}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory consisting of injective objects.

Definition 3.2.18. The additive category $K^+(\mathrm{Proj}(\mathcal{A}))$ is defines as the full subcategory of $\mathrm{K}(\mathcal{A})$ consisting of bounded below chain complexes of projective objects. The morphisms are thus equivalence classes of chain maps modulo chain homotopy.

The additive categories $K^b(\mathrm{Proj}(\mathcal{A})), K^-(\mathrm{Inj}(\mathcal{A})), K^b(\mathrm{Inj}(\mathcal{A}))$ are defined similarly.

There is an apparent functor $K^+(\mathrm{Proj}(\mathcal{A})) \subset \mathrm{K}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$ and there are similar functors for the bounded and negatively bounded variants.

The main result of this section is the following:

Theorem 3.2.19. *Let $\mathcal{A}$ be an abelian category.*

(i) Suppose that $\mathcal{A}$ has enough projecties. Then the functor

$$K^+(\mathrm{Proj}(\mathcal{A})) \to \mathcal{D}^+(\mathcal{A}) \tag{35}$$

is an equivalence of categories.

(ii) Suppose that $\mathcal{A}$ has enough injectives. Then the functors

$$K^-(\mathrm{Inj}(\mathcal{A})) \to \mathcal{D}^-(\mathcal{A})$$

is an equivalence of categories.

Before we can prove Theorem 3.2.19, we need a few preparatory lemmas. The first lemma concerns finding 'projective resolutions' for chain complexes.

Lemma 3.2.20. *Let $\mathcal{A}$ be an abelian category with enough projectives. Let $C_* \in \mathrm{Ch}(\mathcal{A})$ be a bounded below chain complex with $C_m \simeq 0$ for all $m \leq N$.*

(1) There exists a bounded double chain complex $P_{*,*}$ in $\mathcal{A}$ together with a chain map $\epsilon_* \colon P_{*,0} \to C_*$ such that

- *$P_{i,j}$ is projective for all i, j,*
- *$P_{*,j} \simeq 0$ for all $j < 0$, and $P_{i,*} \simeq 0$ for all $i \leq N$,*
- *$\epsilon_n \colon P_{n,*} \to C_n$ is a projective resolution.*

(2) The chain map ϵ_* induces a quasi-isomorphism of chain complexes

$$\mathrm{tot}^{\oplus}(P_{*,*}) \to C_* \,.$$

We can depict the double chain complex and chain complex as follows:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots \arrow[r] & P_{n+1,1} \arrow[r] \arrow[d] & P_{n,1} \arrow[r] \arrow[d] & P_{n-1,1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & P_{n+1,0} \arrow[r] \arrow[d] & P_{n,0} \arrow[r] \arrow[d] & P_{n-1,0} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathcal{C}_{n+1} \arrow[r] & \mathcal{C}_{n} \arrow[r] & \mathcal{C}_{n-1} \arrow[r] & \dots
\end{tikzcd} \tag{36}$$

Proof sketch. For any n choose a projective resolutions $P_*^{B_n} \to B_n(C_*)$ and $P_*^{H_n} \to H_n(C_*)$. By the Horseshoe Lemma 2.2.5 applied to the exact sequence $B_n(C_*) \to Z_n(C_*) \to H_n(C_*)$, we can find a projective resolution $P_*^{Z_n(C_*)} \to Z_n(C_*)$. Applying the Horseshoe Lemma 2.2.5 again to the short exact sequence $Z_n(C_*) \to C_n \to B_{n-1}(C_*)$ yields a projective resolution $\epsilon_n \colon P_*^{C_n} \to C_n$. We set $P_{i,j} = P_j^{C_i}$ and define the vertical differential as $d^v_{i,j} = (-1)^j d^{P_*^{C_i}}$. The horizontal differential is defined as the following composite of the chain maps appearing in the construction of $P_*^{C_n}$:

$$P_*^{C_{n+1}} \to P_*^{B_n} \to P_*^{Z_n} \to P_*^{C_n} \,.$$

One checks via direct computations that $P_{*,*}$ is a double chain complex and that the maps $(\epsilon_n)_n$ define a chain map $P_{*,0} \to C_*$.

For part (2), consider the augmented double chain complex $\tilde{P}_{*,*}$, as depicted in (36). Then each column of $\tilde{P}_{*,*}$ is exact. Lemma 2.2.19 shows that $\mathrm{tot}^{\oplus}(\tilde{P}_{*,*})$ is also exact. There is an apparent degreewise surjective chain map $\tilde{P}_{*,*} \to P_{*,*}$ that after totalization gives rise to a short exact sequence of chain complexes

$$C_*[-1] \to \mathrm{tot}^{\oplus}(\tilde{P}_{*,*}) \to \mathrm{tot}^{\oplus}(P_{*,*}),$$

see also Remark 2.2.21. Inspecting the associated LES and using that $\mathrm{tot}^{\oplus}(\tilde{P}_{*,*})$ is exact, we find that $\mathrm{tot}^{\oplus}(P_{*,*}) \simeq C_*$ is a quasi-isomorphism. □

Lemma 3.2.21. *Let $f_*\colon P_* \to Q_*$ be a quasi-isomorphism between two bounded below chain complexes of projectives. Then f_* is a chain homotopy equivalence.*

Proof. That f_* is a quasi-isomorphism implies that $\mathrm{cone}(f_*)$ is exact. As a bounded below exact chain complex of projectives, the identity of $\mathrm{cone}(f_*)$ is null homotopic (exercise. Hint: show that any bounded below projective exact chain complex is split exact, and conclude using Exercise 1 on Sheet 6).

We have $\mathrm{cone}(f_*)_n = Q_n \oplus P_{n-1}$ with differential

$$d = \begin{pmatrix} d^Q & -f \\ 0 & -d^P \end{pmatrix}.$$

Let H be a null homotopy of $\mathrm{id}_{\mathrm{cone}(f_*)}$. We write $H_n\colon Q_n \oplus P_{n-1} \to Q_{n+1} \oplus P_n$ in components as

$$H_n = \begin{pmatrix} a & b \\ c & e \end{pmatrix}.$$

We have

$$dH = \begin{pmatrix} d^Q & -f \\ 0 & -d^P \end{pmatrix}\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} d^Q a - fc & d^Q b - fe \\ -d^P c & -d^P e \end{pmatrix}$$

and

$$Hd = \begin{pmatrix} a & b \\ c & e \end{pmatrix}\begin{pmatrix} d^Q & -f \\ 0 & -d^P \end{pmatrix} = \begin{pmatrix} ad^Q & -af - bd^P \\ cd^Q & -cf - ed^P \end{pmatrix}$$

Inspecting the identity $dH + Hd = \mathrm{id}_{\mathrm{cone}(f_*)}$ thus yields the following:

- Inspecting the bottom left component yields $c \circ d_Q = d_P \circ c$, meaning that c_* is a chain map.
- Inspecting the top left component yields $ad + da = \mathrm{id}_{Q_*} + fc$, meaning that $-fc$ is homotopic to the identity.
- Inspecting the bottom right component yields $ed + de = -\mathrm{id}_{P_*} - cf$, meaning that $-cf$ is homotopic to the identity.

We thus see that $-c_*$ is a chain homotopy inverse of f_*, showing that f_* is a chain homotopy equivalence.

□

Lemma 3.2.22. *Let $\mathcal{A}$ be an abelian category with enough projectives. Let $P_*, Q_* \in K^+(\mathrm{Proj}(\mathcal{A}))$ and consider a fraction in* $\mathrm{K}^+(\mathcal{A})$

$$\begin{tikzcd} & C_* \arrow[dl, "s"'] \arrow[dr, "f"] & \\ P_* & & Q_* \end{tikzcd} .$$

Then there exists $g\colon P_* \to Q_*$ such that the fraction fs^{-1} is equivalent to the fraction $g(\mathrm{id}_{P_*})^{-1}$ in $K^+(\mathcal{A})$.

Proof. Since $C_* \in \mathrm{K}^+(\mathcal{A})$ is bounded below, we may apply Lemma 3.2.20 to C_* and obtain a bounded double chain complex $P_{*,*}$ of projectives together with a quasi-isomorphism

$$t\colon \tilde{P}_* := \mathrm{tot}^{\oplus}(P_{*,*}) \longrightarrow C_* .$$

We note that $\tilde{P}_*$ is a bounded below complex of projectives, i.e. $\tilde{P}_* \in K^+(\mathrm{Proj}(\mathcal{A}))$.

Since s and t are quasi-isomorphisms, so is the composite $s \circ t$. Now Lemma 3.2.21 shows that $s \circ t$ is a homotopy equivalence. In particular it has an inverse in $K^+(\mathrm{Proj}(\mathcal{A}))$. The desired map is $g = f \circ t \circ (ts)^{-1}$. Finally, the commutative diagram

$$\begin{tikzcd} & C_* \arrow[dl, "s"'] \arrow[dr, "f"] & \\ P_* & \tilde{P}_* \arrow[l, "st"] \arrow[r, "ft"'] \arrow[u, "t"] \arrow[d, "st"'] & Q_* \\ & P_* \arrow[ul, "id"] \arrow[ur, "g"'] & \end{tikzcd}$$

with $sg \in Q$ exhibits the desired equivalence of fractions. $fs^{-1} \sim g(\mathrm{id}_{P_*})^{-1}$. $\square$

Proof of Theorem 3.2.19. We only prove part (i), that is that the functor

$$i\colon K^+(\mathrm{Proj}(\mathcal{A})) \to \mathcal{D}^+(\mathcal{A})$$

is an equivalence. The other statement follows from a dual argument.

We need to show that i is fully faithful and essentially surjective.

The essential surjectivity follows from Lemma 3.2.20: using the notation from the lemma, the coproduct totalization of the bounded projective double chain complex $P_{*,*}$ is a negatively bounded projective chain complex and quasi-isomorphic to C_*.

The functor being full (meaning surjective on Hom sets) amounts to the statement of Lemma 3.2.22.

Next we show that the functor is faithful (meaning injective on Hom sets). Let $f, g\colon P_* \to Q_*$ be morphisms in $K^+(\mathrm{Proj}(\mathcal{A}))$ such that the fractions $f(\mathrm{id}_{P_*})^{-1}$ and $g(\mathrm{id}_{P_*})^{-1}$ are equivalent in $\mathcal{D}^+(\mathcal{A})$. We have to show that $f = g$. We consider a diagram in $\mathrm{K}^+(\mathcal{A})$ exhibiting the equivalence with $C_* \in \mathrm{K}^+(\mathcal{A})$.

$$\begin{tikzcd}
 & P_* \arrow[dl, "id"'] \arrow[dr, "f"] & \\
P_* & C_* \arrow[l, "t_2"'] \arrow[u, "t"] \arrow[r] \arrow[d, "t_3"] & Q_* \\
 & P_* \arrow[ul, "id"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Commutativity shows $t = t_2 = t_3$. In particular we have $f \circ t = g \circ t$. By Lemma 3.2.20 we can choose a complex $\tilde{P}_* \in K^+(\mathrm{Proj}(\mathcal{A}))$ and a quasi-isomorphism $s\colon \tilde{P}_* \to C_*$. Now $t \circ s$ is a homotopy equivalence by Lemma 3.2.21 and hence an isomorphism in $K^+(\mathrm{Proj}(\mathcal{A}))$. Together with $f \circ t \circ s = g \circ t \circ s$ this shows $f = g$ completing the proof. □

Example 3.2.23. Let k be a field. We describe the negatively bounded derived category of the abelian category Vect_k. Since every object of Vect_k is projective, we have

$$\mathcal{D}^+(\mathrm{Vect}_k) \simeq \mathrm{K}^+(\mathrm{Proj}(\mathrm{Vect}_k)) = \mathrm{K}^+(\mathrm{Vect}_k)\,.$$

The negatively bounded derived category thus coincides with the negatively bounded homotopy category. One can also generalize this equivalence to the unbounded variants by a more direct argument, showing that every quasi-isomorphism in $\mathrm{Ch}(\mathrm{Vect}_k)$ is a chain homotopy equivalence.

We denote by $\mathrm{Ch}_0(\mathrm{Vect}_k) \subset \mathrm{Ch}(\mathrm{Vect}_k)$ the full subcategory of chain complexes with vanishing differential. We claim that the functor

$$\mathrm{Ch}_0(\mathrm{Vect}_k) \subset \mathrm{Ch}(\mathrm{Vect}_k) \to \mathrm{K}(\mathrm{Proj}(\mathrm{Vect}_k)) \simeq \mathcal{D}(\mathrm{Vect}_k)$$

is an equivalence of categories. For this, we first note that every chain complex in Vect_k is quasi-isomorphic to its homology, considered as a chain complex with vanishing differential. The functor is thus essentially surjective. We next show that the functor $\mathrm{Ch}_0(\mathrm{Vect}_k) \to \mathrm{K}(\mathrm{Proj}(\mathrm{Vect}_k))$ is fully faithful. Let $f_*\colon C_* \to D_*$ be a chain map where C_*, D_* have vanishing differential. If f_* is null homotopic, there exists $H = (H_n\colon C_n \to D_{n+1})$ with $dH + Hd = f_*$. But since $d = 0$, this implies $f_* = 0$. Thus, only the trivial maps in $\mathrm{Ch}_0(\mathrm{Vect}_k)$ are null homotopic, showing the fully faithfulness.

The above considerations also apply to the derived category of any *semi-simple* abelian category, meaning an abelian category where every short exact sequence splits (and thus every object is projective).

3.3 Derived functors II

When deriving a left exact functor F, we have previously been considering a collection of individual derived functors L_iF defined on the level of abelian categories. In the following, we discuss an upgrade of these into a single derived functor between the derived categories. We then consider the derived Hom between two objects in the derived category as well as adjunctions between derived functors.

Given an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories, it readily extends to a functor $\mathrm{Ch}^+(\mathcal{A}) \to \mathrm{Ch}^+(\mathcal{B})$ between the categories of bounded below chain complexes. Since F is additive it preserves null homotopic chain maps and thus induces a functor $F\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{B})$. We denote the composite functor with the inclusion $\mathrm{K}^+(\mathrm{Proj}(\mathcal{A})) \subset \mathrm{K}^+(\mathcal{A})$ also by F.

Definition 3.3.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and assume that $\mathcal{A}$ has enough projectives[5]. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. The (total) left derived functor of F is defined as the functor

$$\mathbb{L}F\colon \mathcal{D}^+(\mathcal{A}) \overset{(35)}{\simeq} \mathrm{K}^+(\mathrm{Proj}(\mathcal{A})) \xrightarrow{F} \mathrm{K}^+(\mathcal{B}) \to \mathcal{D}^+(\mathcal{B})\,.$$

Remark 3.3.2. Let $F\colon \mathcal{A} \to \mathcal{B}$ be right exact. Given a bounded below chain complex C_* in $\mathcal{A}$, to compute $\mathbb{L}F(C_*)$, one must find a bounded below chain complex of projectives P_* that is quasi-isomorphic to C_*. Then $\mathbb{L}F(C_*) = F(P_*) \in \mathcal{D}^+(\mathcal{B})$.

As explained in Lemma 3.2.20, the chain complex of projectives P_* can be obtained as the totalization of a double complex composed of projectives resolutions of each degree, which are in turn obtained by combining projective resolutions of the boundaries and cycles of C_* via the Horseshoe Lemma.

Example 3.3.3. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor. The induced functor $F\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{B})$ maps quasi-isomorphisms to quasi-isomorphisms[6]. Thus, by the universal property of the localization $\mathcal{D}^+(\mathcal{A}) = \mathrm{K}^+(\mathcal{A})[Q^{-1}]$, F induces a functor

$$F\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})\,.$$

If $\mathcal{A}$ has enough projectives, there exists a natural isomorphism $F \simeq \mathbb{L}F$. On the level of objects, this can be seen as follows. On objects, the composite of $\mathbb{L}F$ with $\mathrm{K}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$ is given by taking a bounded below chain complex C_* and applying F to a chain complex P_* of projectives with a quasi-isomorphism $P_* \to C_*$. But since F is exact, F preserves quasi-isomorphisms and thus $\mathbb{L}F(C_*) = F(P_*) \simeq F(C_*)$ is an isomorphism in $\mathcal{D}^+(\mathcal{B})$.

Remark 3.3.4. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor and suppose that $\mathcal{A}$ has enough projectives. We say that F is of *finite homological dimension* if there exists $N \geq 0$, such that $L_iF \simeq 0$ for all $i \geq N$. In this case, the total derived functor $\mathbb{L}F$ restricts to a functor

$$\mathbb{L}F\colon \mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})$$

between the bounded derived categories. Note that $\mathcal{D}^{\mathrm{b}}(\mathcal{B}) \subset \mathcal{D}^+(\mathcal{B})$ is the full subcategory generated by objects whose homology is bounded above (and automatically below).

[5]For this definition, it suffices to assume that F is additive

[6]This follows from the observation that for a chain complex C_* in $\mathcal{A}$ and F exact, we have $H_iF(C_*) \simeq FH_i(C_*)$.

3.3.1 The derived Hom

The next goal is to construct a $\mathcal{D}^+(\mathrm{Ab})$-valued version of Hom between two objects in $\mathcal{D}^+(\mathcal{A})$. For this, we first consider the $\mathrm{Ch}^+(\mathrm{Ab})$-valued version of the Hom between two bounded chain complexes.

Definition 3.3.5. Let $\mathcal{A}$ be an abelian category and C_*, D_* bounded below chain complexes in $\mathcal{A}$.

(1) The double chain complex $\widetilde{\mathrm{Hom}}(C_*, D_*)$ is defined as[7]

$$\widetilde{\mathrm{Hom}}(C_*, D_*)_{i,j} = \mathrm{Hom}_{\mathcal{A}}(C_{-i}, D_j)$$

$$d^h_{i,j} = (\text{-}) \circ d\colon \mathrm{Hom}(C_{-i}, D_j) \to \mathrm{Hom}(C_{-(i-1)}, D_j)$$

and

$$d^v_{i,j} = (-1)^{i+j+1} d \circ (\text{-})\colon \mathrm{Hom}(C_{-i}, D_j) \to \mathrm{Hom}(C_{-i}, D_{j-1})\,.$$

(2) We define $\mathrm{Hom}_*(C_*, D_*) \in \mathrm{Ch}^+(\mathrm{Ab})$ as the total chain complex

$$\mathrm{Hom}_*(C_*, D_*) \coloneqq \mathrm{tot}\,\widetilde{\mathrm{Hom}}(C_*, D_*)\,.$$

The construction of $\mathrm{Hom}_*(C_*, D_*)$ extends to a functor

$$\mathrm{Hom}_*(\text{-},\text{-})\colon \mathrm{Ch}^+(\mathcal{A})^{\mathrm{op}} \times \mathrm{Ch}^+(\mathcal{A}) \to \mathrm{Ch}^+(\mathrm{Ab})\,.$$

Lemma 3.3.6. *Let C_*, D_* be two bounded below chain complexes in $\mathcal{A}$. Then*

$$Z_0\,\mathrm{Hom}_*(C_*, D_*) \simeq \mathrm{Hom}_{\mathrm{Ch}^+(\mathcal{A})}(C_*, D_*)$$

and

$$H_0\,\mathrm{Hom}_*(C_*, D_*) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(C_*, D_*)\,.$$

Proof. A 0-cycle amounts to a collection of morphisms

$$f_* = (f_i\colon C_i \to D_i)_i \in \bigoplus_{in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(C_i, D_i) = \mathrm{Hom}_0(C_*, D_*)$$

satisfying that $0 = df_*$, meaning that $0 = f_i \circ d + (-1)^{0+0+1} d \circ f_{i+1}$, i.e. f_* is a chain map.

A 1-chain is an element $H \in \bigoplus_{i+j=1} \mathrm{Hom}_{\mathcal{A}}(C_{-i}, D_j) = \bigoplus_i \mathrm{Hom}_{\mathcal{A}}(C_i, D_{i+1})$, and its differential is $dH + Hd$. □

Lemma 3.3.7. *The functor* $\mathrm{Hom}_*(\text{-},\text{-})\colon \mathrm{Ch}^+(\mathcal{A})^{\mathrm{op}} \times \mathrm{Ch}^+(\mathcal{A}) \to \mathrm{Ch}^+(\mathrm{Ab})$ *descends to a functor*

$$\mathrm{Hom}_*(\text{-},\text{-})\colon \mathrm{K}^+(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathrm{Ab})\,.$$

Proof. Exercise. □

[7] A similar definition also appears in the proof of Theorem 2.2.20.

Definition 3.3.8. Suppose that $\mathcal{A}$ has enough projectives. We define the functor

$$\mathrm{RHom}(\text{-},\text{-})\colon \mathcal{D}^+(\mathcal{A})^{\mathrm{op}} \times \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathrm{Ab})$$

as the composite

$$\mathcal{D}^+(\mathcal{A})^{\mathrm{op}} \times \mathcal{D}^+(\mathcal{A}) \overset{(3.2.19)}{\simeq} \mathrm{K}^+(\mathrm{Proj}(\mathcal{A}))^{\mathrm{op}} \times \mathrm{K}^+(\mathrm{Proj}(\mathcal{A})) \subset \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A})$$
$$\xrightarrow{\mathrm{Hom}_*(\text{-},\text{-})} \mathrm{K}^+(\mathrm{Ab}) \to \mathcal{D}^+(\mathrm{Ab})\,.$$

Lemma 3.3.9. *Let $P_* \in \mathrm{K}^+(\mathrm{Proj}(\mathcal{A}))$. Then*

$$\mathrm{Hom}_*(P_*,\text{-})\colon\ \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathrm{Ab})$$

maps quasi-isomorphisms to quasi-isomorphisms.

Proof sketch. Let $f_*\colon C_* \to D_*$ be a quasi-isomorphism. One can show that

$$\mathrm{cone}\,\mathrm{Hom}_*(P_*, f_*) \simeq \mathrm{Hom}_*(P_*, \mathrm{cone}(f_*))\,.$$

Note that $\mathrm{cone}(f_*)$ is an exact chain complex if and only if f_* is a quasi-isomorphism. It thus suffices to show that $\mathrm{Hom}_*(P_*,\text{-})$ preserves exact chain complexes. Let A_* be an exact chain complex. Using Lemma 3.3.6, we have

$$H_n\,\mathrm{Hom}_*(P_*, A_*) \simeq H_0\,\mathrm{Hom}_*(P_*, A_*[-n]) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(P_*, A_*[-n])\,.$$

Using that P_* is projective and $A_*[-n]$ exact, one can show that any chain map $P_* \to A_*[-n]$ is null homotopic (exercise). $\square$

Remark 3.3.10. Lemma 3.3.9 implies that given P_* a projective bounded below chain complex in $\mathcal{A}$ and Q_* any bounded below chain complex in $\mathcal{A}$, we have

$$\mathrm{RHom}(P_*, Q_*) \simeq \mathrm{Hom}_*(P_*, Q_*) \in \mathcal{D}^+(\mathrm{Ab})\,,$$

meaning that it is not necessary to use a projective resolution of Q_*.

Proposition 3.3.11. *Suppose that $\mathcal{A}$ has enough projectives.*

(1) For $C_*, D_* \in \mathcal{D}^+(\mathcal{A})$, we have $H_0\,\mathrm{RHom}(C_*, D_*) \simeq \mathrm{Hom}_{D^+(\mathcal{A})}(C_*, D_*)$.

(2) Let $A, B \in \mathcal{A}$, and consider them as chain complexes concentrated in degree 0. Then

$$H_{-n}\,\mathrm{RHom}(A, B) \simeq \mathrm{Ext}^n_{\mathcal{A}}(A, B)\,.$$

(3) Let $A, B \in \mathcal{A}$, and consider them as chain complexes concentrated in degree 0. Then

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) \simeq \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, B[n])\,.$$

Proof. Part (1) follows from Lemma 3.3.6 and Theorem 3.2.19.

For part (2), let $P_* \to A$ be a projective resolution of A. Then

$$H_{-n}\,\mathrm{RHom}(A,B) \simeq H_{-n}\,\mathrm{Hom}_*(P_*,B) \simeq H^n\,\mathrm{Hom}_{\mathcal{A}}(P_*,B) = \mathrm{Ext}^n_{\mathcal{A}}(A,B)\,,$$

where we pass in the second equivalence from chain complexes and homology to cochain complexes and cohomology.

Part (3) follows from combining parts (1) and (2), using that $H_{-n}\,\mathrm{RHom}(A,B) \simeq H_0\,\mathrm{RHom}(A,B)[n] \simeq H_0\,\mathrm{RHom}(A,B[n])$. □

Corollary 3.3.12. *Suppose that $\mathcal{A}$ has enough projectives. Then the functor $\mathcal{A} \to \mathcal{D}^+(\mathcal{A})$, given by considering each object in $\mathcal{A}$ as a chain complex concentrated in degree 0, is fully faithful.*

We note that the above corollary is also true if $\mathcal{A}$ does not have enough projectives, as can be shown using the calculus of fractions.

3.3.2 Derived adjunctions

Theorem 3.3.13. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{B}$ an abelian category with enough injectives. Let $F\colon \mathcal{A} \leftrightarrow \mathcal{B} : G$ be an adjunction and suppose that F, G are of finite homological dimensions. Then the total derived functors assemble into an adjuntion*

$$\mathbb{L}F\colon \mathcal{D}^b(\mathcal{A}) \longleftrightarrow \mathcal{D}^b(\mathcal{B}) : \mathbb{R}G\,.$$

Furthermore, there exists an isomorphism in $\mathcal{D}^b(\mathrm{Ab})$

$$\mathrm{RHom}(\mathbb{L}F(C_*), D_*) \simeq \mathrm{RHom}(C_*, \mathbb{R}G(D_*))$$

natural in $C_* \in \mathcal{D}^b(\mathcal{A})^{\mathrm{op}}$ and $D_* \in \mathcal{D}^b(\mathcal{B})$.

Proof of Theorem 3.3.13. We first note that the assumptions on $\mathcal{A}, \mathcal{B}$ and F, G guarantee that the derived functors $\mathbb{L}F$ and $\mathbb{R}G$ exist.

Let $C_* \in \mathcal{D}^b(\mathcal{A})$ and $D_* \in \mathcal{D}^b(\mathcal{B})$. The assumptions guarantee there exists an isomorphism $C_* \simeq P_*$ in $\mathcal{D}^+(\mathcal{A})$ with P_* a bounded below chain complex of projectives satisfying that $F(P_*) \in \mathcal{D}^b(\mathcal{B})$. Similarly, we have $D_* \simeq I_*$ in $\mathcal{D}^-(\mathcal{B})$ with I_* a bounded above chain complex of injectives satisfying that $G(I_*) \in \mathcal{D}^b(\mathcal{A})$. For $n \in \mathbb{Z}$ consider the natural isomorphism

$$\Phi\colon\ \mathrm{Hom}_*(F(P_*), I_*)_n = \bigoplus_{i+j=n} \mathrm{Hom}(F(P_{-i}), I_j) \simeq \bigoplus_{i+j=n} \mathrm{Hom}(P_{-i}, G(I_j)) \simeq \mathrm{Hom}_*(P_*, G(I_*))_n\,.$$

This isomorphism is compatible with the differentials on both sides: for $f\colon F(P_{-i}) \to I_j$, we have

$$\Phi(d^h \circ f) = \Phi(f \circ F(d)) = \Phi(f) \circ d = d^h \circ \Phi(f)$$

and

$$\Phi(d^v \circ f) = \Phi(d \circ f) = G(d) \circ \Phi(f) = d^v \circ \Phi(f)$$

by the naturality of Φ, see Remark 1.6.28, meaning that Φ is compatible with both the horizontal and the vertical differential. Thus Φ is also compatible with the differential of the totalizations.

We have thus produced a natural isomorphism in $\mathcal{D}^b(\mathrm{Ab})$

$$\mathrm{Hom}_*(F(P_*), I_*) \simeq \mathrm{Hom}_*(P_*, G(I_*))\,.$$

Using that $\mathbb{L}F(C_*) \simeq F(P_*)$ and $\mathbb{R}G(D_*) \simeq G(I_*)$, we obtain by Remark 3.3.10 that

$$\mathrm{RHom}(\mathbb{L}F(C_*), D_*) \simeq \mathrm{RHom}(F(P_*), D_*) \simeq \mathrm{Hom}_*(F(P_*), I_*)$$

and

$$\mathrm{RHom}(C_*, \mathbb{R}G(D_*)) \simeq \mathrm{RHom}(C_*, G(I_*)) \simeq \mathrm{Hom}_*(P_*, G(I_*))\,.$$

We thus obtain the desired isomorphism (which can be checked to be natural)

$$\mathrm{RHom}(\mathbb{L}F(C_*), D_*) \simeq \mathrm{RHom}(C_*, \mathbb{R}G(D_*))\,.$$

Passing to H_0, we obtain the other desired natural isomorphism in Ab

$$\mathrm{Hom}_{\mathrm{D}^b(\mathcal{B}_*)}(\mathbb{L}F(C_*), D_*) \simeq \mathrm{Hom}_{\mathcal{D}^b(A_*)}(C_*, \mathbb{R}G(D_*))\,.$$

□

Example 3.3.14 (The derived tensor-Hom adjunction)**.** Let R be a commutative ring and $M \in \mathrm{LMod}_R$ an R-module admitting a finite projective resolution. Then

$$\mathrm{RHom}(A \otimes^{\mathbb{L}}_R M, B) \simeq \mathrm{RHom}(A, \mathbb{R}\,\mathrm{Hom}_R(M, B)) \in \mathcal{D}^b(\mathrm{Ab})$$

for all $A, B \in \mathcal{D}^b(\mathrm{LMod}_R)$.

Example 3.3.15. Let k be a field. Consider the adjunction

$$\mathrm{coker}\colon\ \mathrm{LMod}^{\mathrm{fd}}_{kA_2} \simeq \mathrm{Rep}_k(A_2)^{\mathrm{fd}} \longleftrightarrow \mathrm{LMod}^{\mathrm{fd}}_{kA_2} \simeq \mathrm{Rep}_k(A_2)^{\mathrm{fd}} : \mathrm{ker}$$

where $\mathrm{coker}(V_1 \xrightarrow{f} V_2) = (V_2 \to \mathrm{coker}(f_2))$ and $\mathrm{ker}(V_1 \xrightarrow{f} V_2) = (\mathrm{ker}(f) \to V_1)$.

This gives rise to a derived adjunction

$$\mathbb{L}\,\mathrm{coker}\colon \mathcal{D}^b(\mathrm{LMod}^{\mathrm{fd}}_{kA_2}) \longleftrightarrow \mathcal{D}^b(\mathrm{LMod}^{\mathrm{fd}}_{kA_2}) : \mathbb{R}\,\mathrm{ker}\ . \tag{37}$$

Recall that the projective A_2-representations are given by $P_1 = (k \to k)$ and $P_2 = (0 \to k)$ (P_i is generated by paths beginning in i), and the injective A_2-representations are given by $I_1 = (k \to 0)$ and $I_2 = (k \to k)$ (I_i is generated by paths ending in i).

We have

$$\mathrm{coker}(P_1) = I_1, \quad \mathrm{coker}(P_2) = I_2$$

and

$$\mathrm{ker}(I_1) = P_1, \quad \mathrm{ker}(I_2) = P_2\,.$$

Thus, kernel and cokernel define inverse isomorphisms of additive categories:

$$\mathrm{Proj}(\mathrm{LMod}^{\mathrm{fd}}_{kA_2}) \simeq \mathrm{Inj}(\mathrm{LMod}^{\mathrm{fd}}_{kA_2})\,.$$

Passing to chain complexes modulo homotopy, we obtain that

$$\mathrm{coker}\colon K^b(\mathrm{Proj}(\mathrm{LMod}^{\mathrm{fd}}_{kA_2})) \simeq K^b(\mathrm{Inj}(\mathrm{LMod}^{\mathrm{fd}}_{kA_2})) :\!\ker\,.$$

The above functor describe $\mathbb{L}\,\mathrm{coker}$ and $\mathbb{R}\ker$ when composed with the equivalences with $\mathcal{D}^b(\mathrm{LMod}^{\mathrm{fd}}_{kA_2})$. It follows that the derived adjunction (37) is an adjoint equivalence.

As we discuss next, we can also understand the functor coker as the tensor functor with a kA_2-kA_2-bimodule.

The abelian group of endomorphism $\mathrm{Hom}_{\mathrm{LMod}_{kA_2}}(I_1\oplus I_2, I_1\oplus I_2)$ can be equipped with the structure of a k-algebra, where multiplication is the composition of endomorphisms. This algebra is denoted $\mathrm{End}(I_1\oplus I_2)$ and called the endomorphism algebra. We have that $\mathrm{End}(I_1\oplus I_2)\simeq kA_2$: this follows from $\mathrm{LMod}^{\mathrm{fd}}_{kA_2}\simeq(\mathrm{LMod}^{\mathrm{fd}}_{kA_2})^{\mathrm{op}}$ and the observation that $\mathrm{End}(P_1\oplus P_2)\simeq kA_2$ is the path algebra.

Let us denote $T = I_1\oplus I_2$ and $B=\mathrm{End}(T)\simeq kA_2$. We can consider T as a B-kA_2-bimodule, where the right action of $kA_2\simeq(kA_2)^{\mathrm{op}}$ on T is the given action on $I_1\oplus I_2$, and the left action of B on T is by the map $\mathrm{End}(T)\times T\to T, (\phi,t)\mapsto\phi(t)$. We thus obtain a functor

$$(\text{-})\otimes_{kA_2} T\colon\ \mathrm{LMod}_{kA_2}\longrightarrow \mathrm{LMod}_B\simeq\mathrm{LMod}_{kA_2}\,,$$

which one can show to be naturally isomorphic to coker[8]. As a sanity check, we observe that the functors agree on $P_1\oplus P_2 = kA_2$:

$$\mathrm{coker}(P_1\oplus P_2) = I_1\oplus I_2\simeq kA_2\otimes_{kA_2}(I_1\oplus I_2)\simeq(P_1\otimes P_2)\otimes_{kA_2}T\,.$$

It follows that

$$\mathbb{L}\,\mathrm{coker}\simeq(\text{-})\otimes^{\mathbb{L}}_{kA_2}T\,.$$

We next describe a Theorem due to Happel, generalizing the above example, which characterizes when tensoring with a bimodule gives rise to a derived equivalence.

We call a left module T over a k-algebra A finitely generated if there exists a finite subset $\tilde{T}\subset T$, such that $\mathrm{span}_A(\tilde{T}) = \{x\in T | x=\sum_i a_i\cdot t_i \text{ with } a_i\in A, t_i\in\tilde{T}\text{ for all } i\}$ is equal to T.

Theorem 3.3.16 (Happel '87)**.** *Let k be a field and A a finite dimensional k-algebra. Let T be a finitely generated right A-module and $B=\mathrm{End}(T)$ its endomorphism k-algebra. Then the B-A-bimodule T induces an equivalence of bounded derived categories*

$$(\text{-})\otimes^{\mathbb{L}}_A T\colon \mathcal{D}(\mathrm{LMod}_A)\longrightarrow\mathcal{D}(\mathrm{LMod}_B)$$

if and only if the following hold:

[8] The proof of this fact would lead beyond the scope of this lecture. It can be deduced from the Eilenberg–Watts theorem.

- $\mathrm{Ext}^i_{\mathrm{RMod}_A}(T,T) \simeq 0$ *for all* $i > 0$.
- T *admits a finite resolution by finitely generated projective* A*-modules* P_i*:*

$$\cdots \to 0 \to P_n \to \cdots \to P_1 \to P_0 \to T \to 0\,.$$

- A *admits a finite (co-)resolution in terms of direct factors of direct sums of* T*, i.e.*

$$0 \to A \to T_0 \to T_1 \to \cdots \to T_m \to 0$$

where T_i *is a direct summand of* $T^{\oplus N}$ *for some* $N \geq 0$*. This ensures that '*T *generates* $\mathcal{D}^b(\mathrm{LMod}_A)$*'.*

If the above conditions hold, T *is called a tilting module.*

Example 3.3.17. For $T = I_1 \oplus I_2 \in \mathrm{LMod}_{kA_2}$ the conditions of Happel's theorem apply: Since $I_1 \oplus I_2$ is injective, we have $\mathrm{Ext}^i_{\mathrm{RMod}_A}(T,T) \simeq 0$ for all $i > 0$.

The projective resolution of I_1 is

$$0 \to P_2 = [0 \to k] \to P_1 = [k \to k] \to I_1 = [k \to 0] \to 0$$

and the projective resolution of $I_2 = P_1$ is apparent.

Finally, the resolution of P_2 is

$$0 \to P_2 = [0 \to k] \to I_2 = [k \to k] \to I_1 = [k \to 0] \to 0$$

and the resolution of $P_1 = [k \to k] = I_2$ is also apparent.

3.4 Outlook: Triangulated categories and beyond

The derived category of an abelian category is typically not abelian. One should thus ask what what kind of structure it is. The notion of a triangulated category axiomatizes the properties of the derived categories of abelian categories.

Definition 3.4.1 (Puppe '62, Verdier '63)**.** A triangulated category category consists of an additive category $\mathcal{C}$ together with an autoequivalence $[1]\colon \mathcal{C} \to \mathcal{C}$ and a collection of sequences of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]\,,$$

called triangles and written (f,g,h), satisfying the following conditions:

T1) Every morphism f appears in a triangle (f,g,h). The identity id_X appears in the triangle $(\mathrm{id}_X, 0, 0)$. Given a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\simeq"] & Y \arrow[r, "g"] \arrow[d, "\simeq"] & Z \arrow[r, "h"] \arrow[d, "\simeq"] & X[1] \arrow[d, "\simeq"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

with (f,g,h) a triangle, then (f',g',h') is also a triangle.

T2) (Rotation) If (f, g, h) is a triangle, then so are $(g, h, -f[1])$ and $(-h[-1], f, g)$, where $[-1]$ denotes the inverse functor of $[1]$.

T3) Given triangles (f, g, h) and (f', g', h') and morphisms a, b such that $bf = f'a$, then there exists a morphism c such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, dashed, "\exists c"] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

T4) (The octahedral axiom) Given triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z \xrightarrow{k} X[1]$$

$$X \xrightarrow{uv} Z \xrightarrow{l} Y' \xrightarrow{i} X[1]$$

$$Y \xrightarrow{v} Z \xrightarrow{m} X' \xrightarrow{n} X[1]$$

there exists a triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h} Z'[1]$$

such that

$$l = gm, \quad k = nf, \quad h = j[1]i, \quad ig = u[1]n, \quad fj = mv\,.$$

One can arrange these triangles into the shape of an octahedron, hence the name.

Example 3.4.2. The derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is triangulated: the autoequivalence $[1]\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is induced by the shift functor $[1]\colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$. A sequence (f', g', h') is declared to be a triangle if it is isomorphic to a sequence of the form

$$X \xrightarrow{f} Y \xrightarrow{g} \mathrm{cone}(f) \xrightarrow{h} X[1]\,.$$

Note that if $f = \mathrm{id}_X$, then $\mathrm{cone}(\mathrm{id}_X) \simeq 0$ in $\mathcal{D}(\mathcal{A})$, so that $(\mathrm{id}_X, 0, 0)$ is a triangle.

For T2), we note that $X[1] \simeq \mathrm{cone}(Y \to \mathrm{cone}(f))$ (the right side is a shift of a the mapping cylinder). Thus

$$Y \xrightarrow{g} \mathrm{cone}(f) \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is again a triangle.

T3) follows from a version of Lemma 3.1.13: we can find a dotted morphism

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & \mathrm{cone}(f) \arrow[r] \arrow[d, dashed, "\exists"] & X[1] \arrow[d] \\ X' \arrow[r, "f'"] & Y' \arrow[r] & \mathrm{cone}(f') \arrow[r] & X'[1] \end{tikzcd}$$

from any choice of null homotopy of $X \to Y \to Y' \to \mathrm{cone}(f')$. However, this null homotopy is not necessarily unique, and there can be different morphisms between the cones which make the above diagram commute. For instance, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X[1] \arrow[r, "-\mathrm{id}_{X[1]}"] \arrow[d, dashed, "h"] & X[1] \arrow[d] \\ 0 \arrow[r] & Y \arrow[r, "\mathrm{id}_Y"] & Y \arrow[r] & 0 \end{tikzcd}$$

commutes for any choice of morphism h.

Example 3.4.3. The homotopy category $\mathrm{K}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is also triangulated.

The cone in a triangulated category is unique up to (non-unique) isomorphism:

Lemma 3.4.4. *Let $f\colon X \to Y$ be a morphism in a triangulated category $\mathcal{C}$. Then for any two triangles*

$$X \xrightarrow{f} Y \to Z \to X[1]$$

and

$$X \xrightarrow{f} Y \to Z' \to X[1]$$

there exists an isomorphism $Z \simeq Z'$.

Proof idea. T3) guarantees the existence of a morphism between the triangles, containing a morphism $c\colon Z \to Z'$. For any object W, the functor $\mathrm{Hom}_{\mathcal{C}}(W, \text{-})\colon \mathcal{C} \to \mathrm{Ab}$ turns triangles into long exact sequences, and we thus obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(W,X) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(W,Y) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(W,Z) \arrow[r] \arrow[d, "c\circ(\text{-})"] & \mathrm{Hom}(W,X[1]) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(W,Y[1]) \arrow[d, "\mathrm{id}"] \\ \mathrm{Hom}(W,X) \arrow[r] & \mathrm{Hom}(W,Y) \arrow[r] & \mathrm{Hom}(W,Z') \arrow[r] & \mathrm{Hom}(W,X[1]) \arrow[r] & \mathrm{Hom}(W,Y[1]) \end{tikzcd}$$

The 5-Lemma implies that $c \circ (\text{-})$ is an isomorphism for all W, which then implies that c was an isomorphism. □

The discussion conerning T3) in Example 3.4.2 illustrates an structural problem with the notion of a triangulated category: the construction of the cone is not functorial. Morally, the cone wants to be the pushout along 0, but this fails to be true. For instance, the pushout of

$$\begin{tikzcd} k \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}$$

in $\mathcal{D}(\mathrm{Vect}_k)$ vanishes, even though the cone of $k \to 0$ is $k[1]$. If the cone was a pushout, it would also be functorial.

Even with its structural problems and lack of universal properties, the language of derived categories and triangulated categories has been used with remarkable success.

Nevertheless, it turns out that there is a way to fix the problems, by considering *enhanced* triangulated categories, and that their study opens a wealth of new tools. In short, the structural failures result from inverting all chain homotopy equivalences without also keeping track of the involved chain homotopies. In an enhanced triangulated category, we can control the appearing homotopies and this allows to consider a refined version of the pushout, called the homotopy pushout. We will further explore this in the next semester's course on higher category theory.