Seminar: Cluster Algebras, Winter 2025/26, organized by Prof. Dr. Merlin Christ

Last update: October 27, 2025

Cluster algebras are a class commutative algebras equipped with special generating sets called clusters. These clusters are related with each other via a combinatorial rule called mutation. Cluster algebras have been intensely studied since their introduction around 25 years ago. Examples appear in many beautiful contexts, including Lie theory and Teichmüller theory. Their combinatorial nature has allowed for remarkable applications in diverse areas, including representation theory, geometric representation theory, symplectic geometry, scattering amplitudes.

This seminar will focus on introducing the general theory of cluster algebras and interesting examples. The second half of the seminar will concentrate on cluster algebras arising from so-called marked (topological) surfaces, and their relation with Teichmüller theory.

Organization

This will be a joint seminar for bachelor's students (Hauptseminar) and for master's students (Graduate Seminar).

Each talk will be 80 minutes + 10 minutes for questions afterwards.

Place and time

Thursdays 12-14, Room 0.007.

The first talk will be on October 23, 2025. There will be no talk on November 13, 2025.

Prerequisites

A bachelor's course on algebra.

Prior exposure to commutative algebra and real and complex affine algebraic geometry (primarily for the talks 5,6).

For the second half: basic notions of topology or 2-dimensional differential geometry. Definition of a surface (2-dimensional smooth manifold), metrics, embedded submanifolds, homotopies.

List of talks

1. **Quiver mutation.** (October 23) (Klasz)
   Introduce quivers, mutation of quivers. Examples: Dynkin quivers, quiver of a triangulation of an n-gon. Mutation equivalence of quivers, definition and examples of finite mutation type. [FWZ16, Chapters 2.1, 2.2, 2.6]. The quiver applet.

2. **Cluster algebras.** (October 30) (Mora Holl)
   Definition of matrix mutation and seed mutation. Definition of cluster algebra. Examples: A_2-quiver, n-gon, an example not of symmetric type. [FWZ16, Chapters 2.7, 3.1, 3.2]

3. **The Laurent phenomenon.** (November 06) (TBC)
   State the Laurent and positivity phenomena. Sketch part of the proof of the Laurent phenomenon. [FWZ16, Chapter 3.3]

4. **Finite type classification.** (November 20) (Berners)
   Definition of finite type cluster algebra and comparison with finite mutation type. Prove [FWZ17, Prop. 5.9.4]. The classification of finite type cluster algebras [FWZ17, Chapter 5.2]. Explain the proof strategy of [FWZ17, Chapter 5.10].

5. **The Starfish Lemma.** (November 27) (Levchenko)
The two proofs of the Starfish Lemma [FWZ20, Prop. 6.4.1]. Definition of the upper cluster algebra [BFZ05, Chapters 1.1, 1.2]. Cluster tori in the spectrum of the cluster algebra, see for instance [BM24].

6. **The Grassmannian cluster algebra.** (December 04) (Westerlund)
Introduce the Grassmannian, the Plücker coordiantes and the Plücker relations. Describe the quiver of the cluster algebra structure and sketch the proof of Theorem 6.7.8. [FWZ20, Chapter 6.7].

7. **Cluster algebras of marked surfaces.** (December 11) (Guitart Stamatopol)
Introduce marked surfaces, ideal triangulations, taggings, tagged triangulations and the corresponding cluster algebras. [FST08, Chapters 2,4] [FT18, Chapter 5].

8. **The hyperbolic plane.** (December 18) (Carrasco)
Introduce the Poincaré disk mode [Pen12, Chapter 2.1] and the hyperboloid model [Pen12, Chapter 2.2] for the hyperbolic plane. Define horocycles and explain [Pen12, Lem. 2.11]. Introduce the lambda length of two horocycles [Pen12, Chapter 4.1].

9. **The decorated Teichmüller space.** (January 08) (Pecile)
Prove the Ptolemy relation for lambda lengths [Pen12, Cor. 4.16.(a)]. Introduce the decorated Teichmüller space of a marked surface [FT18, Chapter 7]. Discuss Theorem 7.4 of [FT18], see also [Pen12, Thm. 2.5].

10. **The cluster algebra and the decorated Teichmüller space** (January 15) (Cageaoh)
Introduce lambda lengths of tagged arcs [FT18, Chapter 8] and summarize the relation between the cluster algebra and the subalgebra of coordinates on the decorated Teichmüller space generated by the lambda lengths [FT18, Thm. 8.6.].
Make the relation of the spectrum of the cluster algebra and the decorated Teichmüller space explicit in the case of the n-gon, following [FT18, Remark 16.2].

11. **Coefficients for cluster algebras** (January 22) (Avdieiev)
Introduce coefficients of a cluster algebra in a semi-field and compare with the notion of frozen cluster variables [FWZ17, Chapter 3.6]. Define principal and universal coefficients [FZ07]. Discuss [FZ07, Thm. 3.7]. Introduce the exchange graph of clusters in a cluster algebra [FZ07, Def. 4.2] and prove [FZ07, Thm. 4.6] (see also [FWZ17, Cor. 4.3.6, Rem. 4.3.7]).

12. **Coefficients from laminations** (January 29) (Cai)
Introduce laminations of marked surfaces [FT18, Chapter 12] and the corresponding mutation matrices and cluster algebras with coefficients. Introduce the laminated Teichmüller space and laminated lambda lengths [FT18, Chapter 15] and discuss [FT18, Theorem 15.6]. Deduce [FST08, Theorem 5.6].

References

[BFZ05] A. Berenstein, S. Fomin, and A. Zelevinsky. Cluster algebras. III: Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[BM24] J. Beyer and G. Muller. Deep points of cluster algebras. arXiv:2403.15589, 2024.

[FST08] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. Part I: Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[FT18] S. Fomin and D. Thurston. Cluster algebras and triangulated surfaces. Part II: Lambda lengths. *Memoirs of Amer. Math. Soc.*, 225(1223), 2018.

[FWZ16] S. Fomin, L. Williams, and A. Zelevinsky. Introduction to Cluster Algebras. Chapters 1-3. arXiv:1608.05735, 2016.

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[FWZ20] S. Fomin, L. Williams, and A. Zelevinsky. Introduction to Cluster Algebras. Chapter 6. arXiv:2008.09189, 2020.

[FZ07] S. Fomin and A. Zelevinsky. Cluster algebras IV: coefficients. *Compos. Math. 143, 112–164*, 2007.

[Pen12] R. C. Penner. *Decorated Teichmüller theory.* QGM Master Cl. Ser. Zürich: European Mathematical Society (EMS), 2012.