Higher category theory and the (additive) categorification of cluster algebras

Lecture notes based on a mini-course at the IMJ-PRG, winter 2023/24

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Lectures 1,2: Introduction to stable infinity-categories

We introduce stable and presentable k-linear ∞-categories, their relation with dg-categories, the relation of infinity (co)limits with homotopy (co)limits, semiorthogonal decompositions and lax limits

Lecture 3: Constructible sheaves on graphs

We discuss how to describe constructible sheaves on ribbon graphs in terms of functors out of the exit-path category. Such ribbon graphs arise for instance as the spanning graphs of oriented marked surfaces. We will be interested in such sheaves valued in stable infinity-categories.

Lectures 4,5: Applications to cluster algebra categorification

We discuss examples of constructible sheaves of stable infinity-categories, whose global sections describe infinity-categories relevant for the categorification of surface cluster algebras. This includes 3-Calabi-Yau derived categories of (relative) Ginzburg algebras and 2-Calabi-Yau Frobenius extriangulated/exact ∞-categories. The latter are related with the usual cluster categories of the surfaces.

Contents

1 Introduction to stable ∞-categories

We assume that the reader is familiar with the language of simplicial sets and with basic notions from the theory of ∞-categories, as developed in [Lur09]. For introductory treatments, we refer to [Lur, Cis19].

1.1 Stable ∞-categories

Given an ∞-category $\mathcal{C}$ and two objects $x, y \in \mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathcal{S}$ the mapping space, where $\mathcal{S}$ denotes the ∞-category of spaces. An object $x \in \mathcal{C}$ is called a zero object if $\mathrm{Map}_{\mathcal{C}}(x, y)$ and $\mathrm{Map}_{\mathcal{C}}(y, x)$ are contractible spaces for all $y \in \mathcal{C}$. We write $x = 0$ if x is a zero object. Note that the space of zero objects in an ∞-category is either empty or contractible. An ∞-category with a zero object is called pointed.

Definition 1.1. Let $\mathcal{C}$ be a pointed ∞-category. We call $\mathcal{C}$ stable, if

- $\mathcal{C}$ admits all finite limits and colimits and
- a commutative square in $\mathcal{C}$ is pullback if and only if it is pushout.

There are many equivalent definitions of stable ∞-category, the most standard one might be [Lur17, 1.1.1.9], which is equivalent to the above one by [Lur17, 1.1.3.4].

Notation 1.2. We denote a commutative square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ which is both pullback and pushout by a box $\square$ in the center:

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

Such squares are called *biCartesian* squares.

Definition 1.3. We call a commutative square in an ∞-category

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d, "\beta"] \\ 0 \arrow[r] & c \end{tikzcd}$$

- a fiber sequence if it is pullback. In this case, we write $\mathrm{fib}(\beta) = a$ and call it the fiber of β.
- a cofiber sequence if it is pushout. In this case, we write $\mathrm{cof}(\alpha) = c$ and call it the cofiber of α.

Note that if $\mathcal{C}$ is stable, fiber sequences coincide with cofiber sequences. Further, in this case every morphism in $\mathcal{C}$ admits both a fiber and a cofiber, which are each unique up to equivalence.

An important result is that the homotopy 1-category of a stable ∞-category inherits a canonical triangulated structure, such that the fiber and cofiber sequences give rise to distinguished triangles in this triangulated category, see [Lur17, 1.1.2.14]. The cofiber of a morphism is thus mapped to the cone of the morphism in the triangulated homotopy category. We next describe how to turn the passage to the fiber or cofiber into a functor.

Construction 1.4. Let $\mathcal{C}$ be a stable ∞-category. Consider the functor ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$, whose objects correspond to diagrams $a \to b$ in $\mathcal{C}$. We denote by

$$\mathrm{Fun}^{\text{fib-cofib}}(\Delta^1 \times \Delta^1, C) \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$$

the full subcategory consisting of fiber and cofiber sequences, whose objects we depicted as follows.

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b \arrow[d] \\ 0 \arrow[r] & c \end{tikzcd}$$

Pulling back along the inclusion $\Delta^1 \times \{0\} \subset \Delta^1 \times \Delta^1$ defines a functor

$$\mathrm{res}_{a \to b} \colon \mathrm{Fun}^{\text{fib-cofib}}(\Delta^1 \times \Delta^1, C) \to \mathrm{Fun}(\Delta^1, \mathcal{C}),$$

given on objects by

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b \arrow[d] \\ 0 \arrow[r] & c \end{tikzcd} \quad \mapsto \quad a \to b\,.$$

Similarly, pullback along $\{1\} \times \Delta^1 \subset \Delta^1 \times \Delta^1$ defines a functor

$$\mathrm{res}_{b \to c} \colon \mathrm{Fun}^{\text{fib-cofib}}(\Delta^1 \times \Delta^1, C) \to \mathrm{Fun}(\Delta^1, \mathcal{C}),$$

given on objects by

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b \arrow[d] \\ 0 \arrow[r] & c \end{tikzcd} \quad \mapsto \quad b \to c\,.$$

By [Lur09, 4.3.2.15], the functors $\mathrm{res}_{a \to b}$ and $\mathrm{res}_{b \to c}$ are trivial fibrations, and in particular equivalences of ∞-categories. The idea behind this is as follows: any biCartesian square is fully determined by the morphism $a \to b$ (similarly by the morphism $b \to c$), since the other part of the diagram is obtained by first taking a left Kan extension to add the morphism $a \to 0$ to the diagram, and then taking a right Kan extension to complete the pushout square. The result [Lur09, 4.3.2.15] is a very general statement about extending by Kan extensions giving equivalences between ∞-categories of diagrams, and can be quite useful.

Continuing with the construction, we note that any trivial fibration has an inverse (which is even unique up to contractible space of choices), and we define the cofiber functor

$$\mathrm{cof}\colon \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

as the composite of the inverse of $\mathrm{res}_{a \to b}$ with the restriction functor

$$\mathrm{res}_c\colon \operatorname{Fun}^{\text{fib-cofib}}(\Delta^1 \times \Delta^1, C) \to \mathcal{C}$$

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b \arrow[d] \\ 0 \arrow[r] & c \end{tikzcd} \quad \mapsto \quad c$$

to c. The fiber functor is defined similarly.

We next describe the analog of the triangulated shift.

Definition 1.5. Assume that $\mathcal{C}$ has a zero object and admits all finite limits and colimits.

Let $a \in \mathcal{C}$. Consider the (essentially unique) morphism $a \to 0$. We define the *suspension* (or sometimes called shift) of a as $a[1] \coloneqq \mathrm{cof}(a \to 0)$. Similarly, we define the *delooping* of a as $a[-1] \coloneqq \mathrm{fib}(0 \to a)$.

Exercise 1. Assume that $\mathcal{C}$ has a zero object and admits all finite limits and colimits. Then suspension and delooping form functors $[1], [-1] : \mathcal{C} \to \mathcal{C}$. Furthermore, if $\mathcal{C}$ is stable, then these functor are mutually inverse equivalences.

Exercise 2. There exists an equivalence of functors $\mathrm{cof} \simeq \mathrm{fib} \circ [1]$.

Definition 1.6. A functor between stable ∞-categories is called exact, if it preserves fiber and cofiber sequences.

Remark 1.7. Let F be a functor between stable ∞-categories. The following are equivalent:

- F is exact.
- F preserves finite limits.
- F preserves finite colimits.

Exact functors between stable ∞-categories induce triangulated functors on the triangulated homotopy categories.

Example 1.8. Let $\mathcal{C}$ be stable. Then $\operatorname{Fun}(\Delta^1, \mathcal{C})$ is also stable. The functors $[1]\colon \mathcal{C} \to \mathcal{C}$ and $\mathrm{cof}\colon \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ are exact.

Example 1.9. Let A be a ring. There exists a stable ∞-category $\mathcal{D}(A)$, called the unbounded derived ∞-category of right A-modules, or for short the derived ∞-category of A. There are different ways to construct this ∞-category. It arises for instance as the underlying ∞-category, i.e. the ∞-categorical localization $\mathrm{Ch}(A)[\mathrm{quism}^{-1}]$, of the model category of chain complexes of right A-modules with the projective module structure. We will see another constructing passing through dg-algebras further below.

1.2 Presentable ∞-categories

Presentable stable ∞-categories have excellent formal properties, for instance they admit a very simple adjoint functor theorem. Before we state their formal definition, let us briefly illustrate the concept of Ind-completion, see [Lur09, §5.3] for a detailed treatment.

Given a (small) ∞-category $\mathcal{C}$, one can form its Ind-completion $\mathrm{Ind}(\mathcal{C})$, which has a universal property exhibiting it as obtained from $\mathcal{C}$ by freely adding (small) filtered colimits, see [Lur09, 5.3.5.10]. More generally, one can fix a regular cardinal κ and freely add colimits over (small) κ-filtered ∞-categories, obtaining $\mathrm{Ind}_\kappa(\mathcal{C})$. Note that $\mathrm{Ind}_\omega(\mathcal{C}) = \mathrm{Ind}(\mathcal{C})$. Formally, $\mathrm{Ind}_\kappa(\mathcal{C})$ can be defined as a full subcategory of the ∞-category of presheaves on $\mathcal{C}$.

Definition 1.10. An ∞-category $\mathcal{C}$ is called presentable if it admits (small) colimits and is accessible, meaning that there exists a regular cardinal κ and a small ∞-category $\mathcal{D}$, such that $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{D})$.

In many examples of presentable ∞-categories, one encounters the case that $\kappa = \omega$ in above definition.

Example 1.11. Let A be a ring. Then the unbounded derived ∞-category $\mathcal{D}(A)$ is stable and presentable. We have $\mathcal{D}(A) \simeq \mathrm{Ind}\,\mathcal{D}^{\mathrm{perf}}(A)$, and the perfect derived ∞-category $\mathcal{D}^{\mathrm{perf}}(A)$ describes the full subcategory of $\mathcal{D}(A)$ of compact objects.

Remark 1.12. If $\mathcal{C}$ is stable, then $\mathrm{Ind}(\mathcal{C})$ is stable and presentable and $\mathcal{C} \subset \mathrm{Ind}(\mathcal{C})$ is a stable subcategory. Furthermore, if $\mathcal{C}$ is idempotent complete, then $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C})^{\mathrm{c}}$ is equivalent to the full subcategory of $\mathrm{Ind}(\mathcal{C})$ of compact object.

If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between stable ∞-categories, then $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ is a colimits preserving functor between presentable and stable ∞-category. By passing to Ind-completions, we can thus always restrict to Ind-complete stable ∞-categories and colimit preserving functors.

Presentable ∞-categories have many desirable properties. For example, they admit all limits and colimits. The most important one is the following ∞-categorical adjoint functor theorem.

Theorem 1.13 (∞-categorical adjoint functor theorem, [Lur09, 5.5.2.9])**.** *A functor between presentable ∞-categories admits*

- *a right adjoint if and only it preserves (small) colimits and*
- *a left adjoint if and only it preserves (small) limits and is accessible (meaning that it preserves κ-filtered colimits for some regular cardinal κ).*

Example 1.14. Let A, B be two rings and M any A-B-bimodule. There is an exact functor $\text{-} \otimes M \colon \mathcal{D}^{\mathrm{perf}}(A) \to \mathcal{D}(B)$. By the universal property of the Ind-completion, this functor extends to a colimit preserving functor $\text{-} \otimes M \colon \mathcal{D}(A) \to \mathcal{D}(B)$, which thus admits a right adjoint, which we can denote by $\mathrm{RHom}_{\mathcal{D}(B)}(M, \text{-})$.

Definition 1.15. We denote by $\mathcal{P}r^L \subset \mathrm{Cat}_\infty$ the subcategory consisting of presentable ∞-categories and functors which admit a right adjoint. We denote by $\mathcal{P}r^L_{\mathrm{St}} \subset \mathcal{P}r^L$ the full subcategory consisting of stable ∞-categories.

The ∞-category $\mathcal{P}r^L$ admits a symmetric monoidal structure (in the sense of [Lur17]), with monoidal product denoted $\otimes$, satisfying that colimit preserving functors $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$, with $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}r^L$, are in bijection with functors $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ preserving colimits in both entries. If $\mathcal{A}, \mathcal{B} \in \mathcal{P}r^L_{\mathrm{St}}$ are stable, then $\mathcal{A} \otimes \mathcal{B} \in \mathcal{P}r^L_{\mathrm{St}}$ is also stable. A similar monoidal structure on abelian categories is known as the Deligne tensor product.

1.3 k-linear ∞-categories

The theory of symmetric monoidal ∞-categories developed in [Lur17] gives a lift of the usual theory of symmetric monoidal 1-categories that keeps all the usual features the theory. For instance, given a symmetric monoidal ∞-category $\mathcal{C}$, there is a notion of algebra object or commutative algebra object in $\mathcal{C}$.

Example 1.16. The derived ∞-category $\mathcal{D}(k)$ of a field k is symmetric monoidal, and algebra objects in $\mathcal{D}(k)$ can be identified with k-linear dg-algebras.

Example 1.17. Commutative algebra objects in the symmetric monoidal ∞-category $\mathcal{P}r^L_{\mathrm{St}}$ can be identified with monoidal, stable presentable ∞-categories $\mathcal{C}$, satisfying that the monoidal product $\text{-} \otimes \text{-} \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in both entries. Thus, for example, $\mathcal{D}(k) \in \mathcal{P}r^L_{\mathrm{St}}$ gives a commutative algebra object.

Using the general machinery of ∞-categorical modules, also developed in [Lur17], one we can thus consider modules over $\mathcal{D}(k)$ in $\mathcal{P}r^L_{\mathrm{St}}$; we refer to these as k-linear ∞-categories (dropping the words stable and presentable for brevity). We denote the ∞-category of k-linear ∞-categories by

$$\mathrm{LinCat}_k \coloneqq \mathrm{Mod}_{\mathcal{D}(k)}(\mathcal{P}^L_{\mathrm{St}})\,.$$

The module action equips a k-linear ∞-category $\mathcal{C}$ with a tensor product map $\text{-}\otimes\text{-} : \mathcal{D}(k)\times\mathcal{C} \to \mathcal{C}$ which preserves colimits in both entries. Morphisms in LinCat_k are called k-linear functors.

Example 1.18. Let A be a k-linear algebra. Then $\mathcal{D}(A)$ is a k-linear ∞-category. Given two k-linear algebras A, B, k-linear functors $\mathcal{D}(A) \to \mathcal{D}(B)$ can be identified with objects in $\mathcal{D}(A^{\mathrm{op}} \otimes_k B)$.

Example 1.19. As an ∞-category of modules over a commutative algebra object, the ∞-category LinCat_k inherits a canonical symmetric monoidal structure from $\mathcal{P}r^L_{\mathrm{St}}$. Given two rings k-algebras A, B, we have $\mathcal{D}(A \otimes_k B) \simeq \mathcal{D}(A) \otimes_{\mathrm{LinCat}_k} \mathcal{D}(B)$.

k-linear ∞-categories come with a k-linear Hom, referred to as the morphism object:

Definition 1.20 ([Lur17, 4.2.1.28])**.** Let $\mathcal{C}$ be a k-linear ∞-category and $X, Y \in \mathcal{C}$. A morphism object $\mathrm{Mor}_{\mathcal{C}}(X,Y) \in \mathcal{D}(k)$ is a k-module equipped with a morphism in $\mathcal{C}$

$$\alpha : \mathrm{Mor}_{\mathcal{C}}(X,Y) \otimes X \to Y,$$

such that for every object $C \in \mathcal{D}(k)$ composition with α induces an equivalence of spaces

$$\mathrm{Map}_{\mathcal{D}(k)}(C, \mathrm{Mor}_{\mathcal{C}}(X,Y)) \xrightarrow{\text{-}\otimes X} \mathrm{Map}_{\mathcal{C}}(C \otimes X, \mathrm{Mor}_{\mathcal{C}}(X,Y) \otimes X) \longrightarrow \mathrm{Map}_{\mathcal{C}}(C \otimes X, Y)\,.$$

Morphism objects in k-linear ∞-categories always exist and satisfy

$$\mathrm{Ext}^{-i}(X,Y) \coloneqq \mathrm{H}_i\,\mathrm{Mor}_{\mathcal{C}}(X,Y) \simeq \pi_0\,\mathrm{Map}_{\mathcal{C}}(X[i],Y)\,.$$

The formation of morphism objects forms a functor

$$\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-}) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}(k)$$

which preserves limits in both entries. For $X \in \mathcal{C}$, the functor $\mathrm{Mor}_{\mathcal{C}}(X,\text{-})$ is right adjoint to the functor $\text{-} \otimes X : \mathcal{D}(k) \to \mathcal{C}$.

1.4 From dg-categories to k-linear ∞-categories

Let C be a k-linear dg-category. The dg-nerve $\mathrm{N}_{\mathrm{dg}}(C) \in \mathrm{Cat}_\infty$ of C is an explicitly defined ∞-category, see [Lur17, Section 1.3.1], with

- objects the objects in C,
- morphisms the 0-cycles in C,
- 2-simplicies consisting of three 0-cycles $f: a \to b$, $g: b \to c$ and $h: a \to c$, together with a 1-chain q in the chain complex $C(a,c)$, satisfying $dq = (g \circ f) - h$.

We denote by dgMod_C the dg-category of right dg C-modules, meaning dg-functors $C^{\mathrm{op}} \to \mathrm{Ch}(k)$. Passing to the dg-nerve of dgMod_C corresponds to taking the ∞-categorical localization at (only) the chain homotopy equivalences. To obtain the derived ∞-category, we do the following (compare also with Whitehead's theorem for model categories).

Definition 1.21. The unbounded derived ∞-category $\mathcal{D}(C)$ of C is defined as the dg-nerve of the full dg subcategory $\mathrm{dgMod}_C^{\mathrm{cf}} \subset \mathrm{dgMod}_C$ consisting of cofibrant dg C-modules (all objects are fibrant) with respect to the projective model structure (defined e.g. in [Toë07]).

Remark 1.22. Let C be a dg-category. The weak equivalences W in the projective model structure on dgMod_C are given by the quasi-isomorphisms. It's not too hard to show that the derived ∞-category $\mathcal{D}(C)$ has the universal property of the localization[1] $\mathrm{dgMod}_C[W^{-1}]$. If C is a (discrete) ring, this is proven in [Lur17, Section 1.3]. A generalization of the proof for C a dg-algebra (dg-category with a single object), is described in [Chr22b, Section 2.4].

We will show below that $\mathcal{D}(C)$ is stable. It is further a presentable and k-linear ∞-category.

Example 1.23. Let $f : A \to B$ be a morphism between dg-algebras. This endows B with the structure of a dg A-module. The tensor dg-functor

$$- \otimes_A B : \mathrm{dgMod}_A \longrightarrow \mathrm{dgMod}_B \tag{1}$$

preserves cofibrant objects (since its the left adjoint in a Quillen adjunction) and thus restricts to a functor

$$- \otimes_A B : \mathrm{dgMod}_A^{\mathrm{cf}} \longrightarrow \mathrm{dgMod}_B^{\mathrm{cf}}\,.$$

Passing to dg-nerves, we obtain a colimit preserving functor between stable, presentable ∞-categories, denoted

$$f_! : \mathcal{D}(A) \longleftrightarrow \mathcal{D}(B)\,. \tag{2}$$

One can also write $f_! = - \otimes_A^L B = - \otimes_A B$. The functor $f_!$ admits a right adjoint $f^* = \mathrm{RHom}_B(B,-)$, which is induced from the dg-functor $\mathrm{Hom}_B(B,-) : \mathrm{dgMod}_B \to \mathrm{dgMod}_A$.

The adjointness of $f_!$ and f^* follows from the fact that $- \otimes_A B \dashv \mathrm{Hom}_B(B,-)$ is a Quillen adjunction and the fact it thus induces an adjunction on the level of the underlying ∞-categories, see [MG16, Cis19].

The above story can be generalized to dg-categories and dg-bimodules. A dg A-B-bimodule is a right dg $A^{\mathrm{op}} \otimes B$-module. Let $M \in \mathrm{dgMod}_{A^{\mathrm{op}} \otimes B}$ be cofibrant. The dg-nerve of the dg-functor $- \otimes_A M : \mathrm{dgMod}_A^{\mathrm{cf}} \to \mathrm{dgMod}_B^{\mathrm{cf}}$ is a k-linear functor $\mathrm{N}_{\mathrm{dg}}(- \otimes_A M) : \mathcal{D}(A) \to \mathcal{D}(B)$ between k-linear ∞-categories.

[1] The ∞-category underlying a model category D with weak equivalences W is defined as the ∞-categorical localization $D[W^{-1}]$.

1.5 ∞-categorical (co)limits via homotopy (co)limits

We are interested in being able to compute two kinds of (∞-categorical) limits and colimits:

1) Limits and colimits in the derived ∞-category $\mathcal{D}(C)$ of a dg-category C.

2) Colimits in the ∞-category LinCat_k of stable, presentable k-linear ∞-categories.

Under mild assumptions, homotopy (co)limits in a model category indexed by 1-categories give rise to ∞-categorical (co)limits in its underlying ∞-category, see Remark 7.9.10 in [Cis19]. As we explain below, we can often compute both of these kinds of (co)limits in terms of homotopy (co)limits. To explicitly compute homotopy (co)limits, one (co)fibrantly replaces the diagram in question and simply computes the 1-categorical (co)limit of the replacement.

1) Limits and colimits in the derived ∞-category $\mathcal{D}(C)$.

Let C be a dg-category. Then dgMod_C is a model category with the projective model structure. The cone of a morphism describes the homotopy pushout of the morphism along 0, and similarly the cocone describes the homotopy pullback of the morphism along 0:

Lemma 1.24. *Let C be a dg-category. Let $\alpha : a \to b$ be a morphism between cofibrant objects in dgMod_C. Let $\mathrm{cone}(\alpha)$ be the apparent dg C-module with underlying chain complex $\left(a[1] \oplus b, d = \begin{pmatrix} -d_a & 0 \\ -\alpha & d_b \end{pmatrix}\right)$. There is a homotopy pushout diagram*

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d] \\ 0 \arrow[r] & \mathrm{cone}(\alpha) \end{tikzcd}$$

and a homotopy pullback diagram

$$\begin{tikzcd} \mathrm{cone}(\alpha)[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ a \arrow[r, "\alpha"] & b \end{tikzcd} .$$

These in turn thus give rise to a cofiber sequence and a fiber sequence in $\mathcal{D}(C)$.

Proof. The diagram

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \\ 0 & \end{tikzcd}$$

is not cofibrant with respect to the projective model structure on the category of such diagrams in dgMod_C. A cofibrant replacement is given by,

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & (a \oplus a[1] \oplus b, d) \\ 0 & \end{tikzcd}$$

where $(a \oplus a[1] \oplus b, d)$ denotes the mapping cylinder of α, with differential d given by $\begin{pmatrix} d_a & \mathrm{id}_a & 0 \\ 0 & -d_a & 0 \\ 0 & -\alpha & d_b \end{pmatrix}$.

This follows from [Lur09, A.2.4.4], stating that for pushouts, it suffices that the three objects are cofibrant and one of the two morphisms is a cofibration. The homotopy pushout is given by the pushout of the above diagram, which in turn coincides with the cone of α. $\square$

Using Lemma 1.24, we can deduce that $\mathcal{D}(C)$ is stable:

Proposition 1.25. *Let C be a dg-category. Then $\mathcal{D}(C)$ is a stable ∞-category.*

Proof. Let $\alpha\colon a \to b$ be a morphism in $\mathrm{dgMod}_{\mathcal{C}}^{\mathrm{cf}}$. Then there is an apparent quasi-isomorphism

$$a[1] \longrightarrow \mathrm{cone}(b \to \mathrm{cone}(\alpha))\,.$$

In particular, we find that the homotopy cofiber sequence

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d] \\ 0 \arrow[r] & \mathrm{cone}(\alpha) \end{tikzcd}$$

is quasi-isomorphic to the homotopy fiber sequence

$$\begin{tikzcd} \mathrm{cone}(b \to \mathrm{cone}(\alpha))[-1] \arrow[r] \arrow[d] & b \arrow[d] \\ 0 \arrow[r] & \mathrm{cone}(\alpha) \end{tikzcd}$$

and thus itself a homotopy fiber sequence. This shows that homotopy fiber sequences and homotopy cofiber sequences in dgMod_C coincide. Hence, fiber sequences and cofiber sequences in $\mathcal{D}(C)$ exist and also coincide.

Since dgMod_C has a homotopy 0-object, $\mathcal{D}(C)$ is furthermore pointed. It thus follows that $\mathcal{D}(C)$ is stable (using the usual definition of stability of [Lur17, 1.1.1.9]). $\square$

Remark 1.26. Proposition 1.25 is a special case of the more general statement that the dg-nerve of any pretriangulated dg-category is stable, see [Fao17, Thm. 4.3.1].

2) Colimits in LinCat_k

Recall that:

- a quasi-equivalence of dg-categories is a dg-functor $A \to B$, which defines an equivalences on the homotopy categories and quasi-isomorphisms on the morphism chain complexes.
- a Morita equivalence is a dg-functor $A \to B$ giving rise to a quasi-equivalence $\mathrm{dgMod}_A^{\mathrm{perf}} \to \mathrm{dgMod}_B^{\mathrm{perf}}$.

There are two model structures on the category of dg-categories dgCat_k, one with weak equivalences the collection Q of quasi-equivalences and one with weak equivalences the collection M of Morita equivalences. The latter model structure is a left Bousfield localization of the former, so that the induced functor $\mathrm{dgCat}_k[Q^{-1}] \to \mathrm{dgCat}_k[M^{-1}]$ preserves homotopy colimits.

The passage to the derived ∞-category defines an equivalence

$$\mathcal{D}(\text{-})\colon\ \mathrm{dgCat}_k[M^{-1}] \simeq \mathrm{LinCat}_k^{\mathrm{cpt\text{-}gen}}$$

between the ∞-category underlying dgCat_k with the Morita model structure and the subcategory $\mathrm{LinCat}_k^{\mathrm{cpt\text{-}gen}} \subset \mathrm{LinCat}_k$ of compactly generated[2] ∞-categories and compact objects preserving functors, see [Coh13]. Thus homotopy colimit diagrams indexed by 1-categories in dgCat_k give rise to ∞-categorical colimit diagrams in LinCat_k.

We will compute some examples later on.

[2]Compactly geneated meaning ∞-categories $\mathcal{C}$ satisfying that $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^c)$.

1.6 Semiorthogonal decompositions

There are two main constructions, which are referred to as 'gluing' of stable ∞-categories. The first is taking limits or colimits of certain diagrams of stable ∞-categories, for instance arising from constructible sheaves, see the next section. Another way is to glue stable ∞-categories along functors to produce semiorthogonal decompositions.

Definition 1.27. Let $\mathcal{V}$ be a stable ∞-categories. We call a full subcategory $\mathcal{A} \subset \mathcal{V}$ a stable subcategory if $\mathcal{A}$ is stable, the inclusion functor is exact and its image is closed under equivalences.

Definition 1.28. Let $\mathcal{V}$ be a stable ∞-category and let $\mathcal{A}, \mathcal{B}$ be stable subcategories of $\mathcal{V}$. We call the pair $(\mathcal{A}, \mathcal{B})$ a semiorthogonal decomposition (of length 2) of $\mathcal{V}$ if

1) for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the mapping space $\mathrm{Map}_{\mathcal{V}}(b, a)$ is contractible and

2) for every $x \in \mathcal{V}$, there exists a fiber and cofiber sequence $b \to x \to a$ in $\mathcal{V}$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

An easy way to define semiorthogonal decompositions of length $n \geq 3$ is recursively in terms of semiorthogonal decomposition of length 2.

Example 1.29. Let $\mathcal{C}$ be a stable ∞-category and $\mathcal{V} = \mathrm{Fun}(\Delta^1, \mathcal{C})$. Let $\mathcal{A} \simeq \mathcal{C}$ be the full subcategory of $\mathcal{V}$ spanned by diagrams of the form $x \to 0$ and $\mathcal{B} \simeq \mathcal{C}$ the full subcategory of $\mathcal{V}$ spanned by diagrams of the form $0 \to y$. Then $(\mathcal{A}, \mathcal{B})$ forms a semiorthogonal decomposition of $\mathcal{V}$.

Similarly, $\mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$ has a canonical semiorthogonal decomposition of length n.

Definition 1.30. Let $\mathcal{V}$ be a stable ∞-category and let $i : \mathcal{A} \to \mathcal{V}$ be the inclusion of a stable subcategory. We call $\mathcal{A}$ admissible if i admits both left and right adjoints.

Remark 1.31. Given a stable subcategory $\mathcal{A} \subset \mathcal{V}$, the right orthogonal $\mathcal{A}^{\perp} \subset \mathcal{V}$ is the full subcategory spanned by objects $x \in \mathcal{V}$, such that the mapping space $\mathrm{Map}_{\mathcal{V}}(a, x)$ is contractible for all $a \in \mathcal{A}$. The left orthogonal ${}^{\perp}\mathcal{A}$ is defined similarly. One can show the following, see for instance [DKSS21]:

- If $(\mathcal{A}, \mathcal{B})$ forms a semiorthogonal decomposition, then $\mathcal{B} = {}^{\perp}\mathcal{A}$ and $\mathcal{A} = \mathcal{B}^{\perp}$.
- $(\mathcal{A}, \mathcal{A}^{\perp})$ forms a semiorthogonal decomposition if and only if $\mathcal{A} \subset \mathcal{V}$ admits a left adjoint and $(\mathcal{A}^{\perp}, \mathcal{A})$ forms a semiorthogonal decomposition if and only if $\mathcal{A} \subset \mathcal{V}$ admits a right adjoint.
- Thus, $\mathcal{A} \subset \mathcal{V}$ is admissible if and only if $(\mathcal{A}, {}^{\perp}\mathcal{A})$ and $(\mathcal{A}^{\perp}, \mathcal{A})$ form semiorthogonal decompositions of $\mathcal{V}$.

Remark 1.32. Let $(\mathcal{A}, \mathcal{B})$ be a semiorthogonal decomposition of $\mathcal{V}$. Let $\mathcal{V} \to \mathcal{B}$ be right adjoint adjoint of the inclusion $\mathcal{B} \subset \mathcal{V}$. Then the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ 0 \arrow[r] & \mathcal{B} \end{tikzcd} \tag{3}$$

forms a fiber and cofiber sequence in the ∞-category St of stable ∞-categories and exact functors. In this case, $\mathcal{B}$ is also known as the Verdier quotient of $\mathcal{V}$ by $\mathcal{A}$.

Conversely, consider any diagram of the form (3). If the diagram defines a fiber and cofiber sequence in St and $\mathcal{V} \to \mathcal{B}$ admits a fully faithful right adjoint, then $(\mathcal{A}, \mathcal{B})$ forms a semiorthogonal decomposition of $\mathcal{V}$.

To recover $\mathcal{V}$ from the components of the semiorthogonal decompositions, we can ask for the extra data of a *gluing functor*.

Definition 1.33. Let $(\mathcal{A}, \mathcal{B})$ be a semiorthogonal decomposition of $\mathcal{V}$ with $\mathcal{A}$ admissible. Let $j\colon \mathcal{V} \to \mathcal{A}$ be the right adjoint of the inclusion $\mathcal{A} \subset \mathcal{V}$. The gluing functor $F\colon \mathcal{B} \to \mathcal{A}$ of $(\mathcal{A}, \mathcal{B})$ is defined as the composite functor $\mathcal{B} \subset \mathcal{V} \xrightarrow{j} \mathcal{A}$.

Note that we have by adjunction

$$\mathrm{Map}_{\mathcal{A}}(a, F(b)) \simeq \mathrm{Map}_{\mathcal{V}}(a, b) \tag{4}$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Example 1.34. We describe the gluing functors in the semiorthogonal decomposition of Example 1.29. The right adjoint of the inclusion

$$\mathcal{A} = \{x \to 0\} \subset \mathrm{Fun}(\Delta^1, \mathcal{C})$$

is given on objects by the assignment

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \simeq \mathcal{A}, \quad (x \xrightarrow{\alpha} y) \mapsto \mathrm{fib}(\alpha)\,.$$

The restriction to $\mathcal{B}$ is thus given by the functor

$$F\colon \mathcal{B} \simeq \mathcal{C} \simeq \mathcal{A}\,,$$

where we use the identifications $(0 \to y) \in \mathcal{B} \mapsto \mathrm{fib}(0 \to y) \in \mathcal{C}$ and $x \in \mathcal{C} \mapsto (x \to 0) \in \mathcal{A}$.

If $(\mathcal{A}, \mathcal{B})$ has a gluing functor, then $\mathcal{V}$ is the *lax limit* of the gluing functor, considered as a diagram $\Delta^1 \to \mathcal{S}t$. This is an $(\infty, 2)$-categorical universal property in an $(\infty, 2)$-categorical version $\mathcal{S}t$ of the ∞-category of stable ∞-category St. This offers an exciting new perspective on the rather classical concept of semiorthogonal decomposition. We won't describe any details here, but feel that it would have been a loss not to mention this perspective.

Recall that the limit of a diagram is the universal (commuting) cone over the diagram. A lax cone over a diagram in an $(\infty, 2)$-category is a generalization of a cone where the diagram does not have to commute, instead there are non-invertible 2-morphisms. For instance a lax cone over a diagram

$$\Delta^1 \to \mathrm{St}, \quad 0 \to 1 \;\mapsto\; \mathcal{B} \xrightarrow{F} \mathcal{A}$$

amounts to a diagram:

$$\begin{tikzcd} & \mathcal{C} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Nearrow"] & \\ \mathcal{B} \arrow[rr, "F"'] & {} & \mathcal{A} \end{tikzcd}$$

Switching the direction of the non-invertible 2-morphisms changes from lax cones to oplax cones.

The lax limit is defined as the universal lax cone, and determined uniquely up to equivalence of ∞-categories.

If $(\mathcal{A}, \mathcal{B})$ forms a semiorthogonal decomposition of $\mathcal{V}$ with gluing functor F, the lax limit cone is of the form

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\Nearrow"] & \\ \mathcal{B} \arrow[rr, "F"'] & {} & \mathcal{A} \end{tikzcd}$$

where an object $x = \mathrm{cof}(a \to b) \in \mathcal{V}$ with $a \in \mathcal{A}$, $b \in \mathcal{B}$ is mapped to $\phi(x) = b$ and $\psi(x) = \mathrm{cof}(a \to F(b))$, where the morphism $a \to F(b)$ arises from (4). The natural transformation evaluates at x to the apparent cofiber morphism $F(b) \to \mathrm{cof}(a \to F(b))$.

Remark 1.35. Let $F\colon \mathcal{B} \to \mathcal{A}$ be an exact functor. The lax limit of F in St can explicitly be computed as the pullback of the following diagram in St:

$$\begin{tikzcd} & \mathrm{Fun}(\Delta^1, \mathcal{A}) \arrow[d, "\mathrm{ev}_1"] & (a_0 \to a_1) \arrow[d, mapsto] \\ \mathcal{B} \arrow[r, "F"] & \mathcal{A} & a_1 \end{tikzcd}$$

2 Constructible sheaves of stable ∞-categories on graphs

2.1 Constructible sheaves via the exit path ∞-category

A stratified space consists, roughly, of a topological space X together with nested subspaces[3] $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$, subject to certain conditions which depend on the context of interest (e.g. complex stratification, conical stratification, ...). We refer to the complements $\{X_i \backslash X_{i-1}\}$ as the strata of the stratification.

Given a stratified space X, there is an associated simplicial set $\mathrm{Exit}(X)$, whose 0-simplicies are the points of X and whose 1-simplicies are those paths in X which, if they do not remain within one single stratum, only exit a stratum to a stratum of a higher index. Formally, $\mathrm{Exit}(A)$ can be defined as a simplicial subset of the singular simplicial set $\mathrm{Sing}(X)$ of X, see [Lur17, A.6.2], it is called the exit path ∞-category of X. We note that $\mathrm{Exit}(X)$ is indeed an ∞-category if the stratification is conical [Lur17, A.6.4].

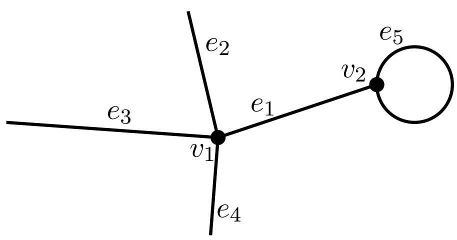
Let us consider the case where $X = \mathbf{G}$ is a graph, with one stratum the vertices and the other stratum the complement of the vertices in $\mathbf{G}$ (i.e. the edges with vertices removed). In this case, the definition exit path ∞-category $\mathrm{Exit}(\mathbf{G})$ is equivalent to the following, very simple definition.

Definition 2.1. Let $\mathbf{G}$ be a graph. We define the exit path ∞-category $\mathrm{Exit}(\mathbf{G})$ as the nerve of the 1-category with

- objects the vertices and edges of $\mathbf{G}$.
- a non-identity morphism $v \to e$ going from a vertex v to an edge e for every endpoint of e at v.

If e is a loop at a vertex v, then there are two morphisms $v \to e$. Note that if there are no loops, the above 1-category is a poset.

Example 2.2. On the left: an example of a graph with two vertices and five edges. Three of the edges are external edges, meaning they are incident to a single vertex. One of the edges is a loop. On the right: its exit path category.



$$\mathrm{Exit}(\Gamma) = \begin{tikzcd} & e_2 & & v_2 \arrow[dl] \arrow[d, bend right] \arrow[d, bend left] \\ e_3 & v_1 \arrow[l] \arrow[u] \arrow[r] \arrow[d] & e_1 & e_5 \\ & e_4 & & \end{tikzcd}$$

We next discuss how to describe constructible sheaves on X using $\mathrm{Exit}(X)$.

[3] More generally, the subspaces can also be indexed by a poset.

Definition 2.3. A sheaf F (with values in some ∞-category $\mathcal{C}$) on a stratified space X is called constructible if $F|_{X_i \setminus X_{i-1}}$ is a locally constant sheaf for all i.

Theorem 2.4 ([Lur17] for $\mathcal{C} = \mathcal{S}$, [Tan19], [PT22])**.** *Let X be a sufficiently nice conically stratified space and $\mathcal{C}$ a compactly generated presentable ∞-category. Then there exists an equivalence of ∞-categories*

$$\mathrm{Shv}_{\mathrm{c}}(X, \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C})$$

between the ∞-category of $\mathcal{C}$-valued constructible sheaves on X and the ∞-category functors from the exit path ∞-category of X to $\mathcal{C}$.

In terms of functors out of the exit path ∞-category, the global sections of a constructible sheaf are obtained as follows:

Definition 2.5. Let $\mathcal{F} : \mathrm{Exit}(X) \to \mathcal{C}$ be a constructible sheaf on X. The global sections $\mathcal{H}(X, \mathcal{F}) \in \mathcal{C}$ of $\mathcal{F}$ are defined as the limit of $\mathcal{F}$.

2.2 Limits and colimits in ∞-categories of ∞-categories

Let us consider the following flavors of ∞-categories of ∞-categories:

- Cat_∞, the ∞-category of ∞-categories.
- St, the subcategory of Cat_∞ of stable ∞-categories and exact functors.
- $\mathcal{P}r^L$, the subcategory of Cat_∞ of presentable ∞-categories and colimit preserving (=left adjoint) functors.
- $\mathcal{P}r^R$, the subcategory of Cat_∞ of presentable ∞-categories and limit preserving and accessible (=right adjoint) functors.
- $\mathcal{P}r^L_{\mathrm{St}} \subset \mathcal{P}r^L$ and $\mathcal{P}r^R_{\mathrm{St}} \subset \mathcal{P}r^R$, the full subcategories consisting of stable and presentable ∞-categories.
- $\mathrm{LinCat}_k = \mathrm{Mod}_{\mathcal{D}(k)}(\mathcal{P}^L_{\mathrm{St}})$ the ∞-category of k-linear ∞-categories.

The forgetful functors can be organized in a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{LinCat}_k \arrow[r] & \mathcal{P}r^L_{\mathrm{St}} \arrow[r] \arrow[drr] & \mathcal{P}r^L \arrow[r] & \mathrm{Cat}_\infty & \mathcal{P}r^R \arrow[l] & \mathcal{P}r^R_{\mathrm{St}} \arrow[l] \arrow[dll] \\
& & & \mathrm{St} \arrow[u] & &
\end{tikzcd} \tag{5}$$

All of the above functor preserve limits. The functors $\mathrm{LinCat}_k \to \mathcal{P}r^L_{\mathrm{St}} \to \mathcal{P}r^L$ furthermore also preserve colimits.

Exercise 3. Find references in [Lur09, Lur17] that the above functors preserve limits.

Hint/comment: I am not aware of a direct reference for the statement that the functor $\mathrm{St} \to \mathrm{Cat}_\infty$ *preserves limits, so don't worry about this one. One can deduce this statement from the explicit description of limits in* Cat_∞ *in terms of sections of the Grothendieck construction, see also below.*

To compute limits in terms of colimits and vice versa in $\mathcal{P}r^L$ and $\mathcal{P}r^R$, we can use the following result:

Theorem 2.6 ([Lur09, 5.5.3.4]). *There are inverse equivalences of ∞-categories*

$$\mathrm{radj} : \mathcal{P}r^L \longleftrightarrow \left(\mathcal{P}r^R\right)^{\mathrm{op}} : \mathrm{ladj}\,,$$

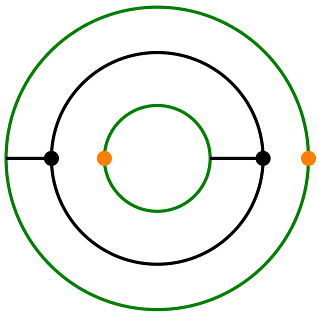
where radj *is the identity on objects and maps a functor to its right adjoint. Similarly,* ladj *maps a functor to its left adjoint.*

Corollary 2.7. *The colimit of a diagram $Z \to \mathcal{P}r^L$ is equivalent to the limit of the right adjoint diagram $Z^{\mathrm{op}} \to \mathcal{P}r^R$.*

The equivalence from Theorem 2.6 clearly restricts to an equivalence between $\mathcal{P}r^L_{\mathrm{St}}$ and $(\mathcal{P}r^R_{\mathrm{St}})^{\mathrm{op}}$. It follows that Corollary 2.7 also holds for limits and colimits in $\mathcal{P}r^L_{\mathrm{St}}$ and $\mathcal{P}r^R_{\mathrm{St}}$.

We already know how to compute colimits in LinCat_k via homotopy colimits of dg-categories. The remainder of this subsections puts the theory into action in a first example:

Example 2.8. Consider the following graph $\mathbf{G}$, which we conveniently embed into an annulus decorated with two orange points (marked points) on the boundary.



We define a constructible sheaf $\mathcal{F}$ on $\mathbf{G}$ via the following diagram in LinCat_k:

$$\begin{tikzcd}
 & & \mathcal{D}(k) & \\
\mathcal{D}(k) & \mathcal{D}(kA_2) \arrow[l, "\varrho_2"'] \arrow[ur, "\varrho_1"] \arrow[dr, "\varrho_3[-1]"'] & \mathcal{D}(k) & \mathcal{D}(kA_2) \arrow[l, "\varrho_1"'] \arrow[ul, "\varrho_1"'] \arrow[dl, "\varrho_3[-1]"] \\
 & & \mathcal{D}(k) &
\end{tikzcd} \tag{6}$$

Here $\mathcal{D}(kA_2)$ is the derived ∞-category of the k-linear path algebra of the A_2-quiver $\cdot \to \cdot$.

Exercise 4. Show that $\mathcal{D}(kA_2) \simeq \mathrm{Fun}(\Delta^1, \mathcal{D}(k))$.
Hint: a natural transformation of functors $\mathcal{D}(kA_2) \to \mathcal{D}(k)$ is by definition a functor $\Delta^1 \times \mathcal{D}(kA_2) \to \mathcal{D}(k)$.

Further,

- the functor ϱ_1 is defined as the right adjoint of the tensor functor

$$(\text{-}) \otimes_k (0 \to k)\colon \mathcal{D}(k) \to \mathcal{D}(kA_2)\,.$$

  Explicitly, we have that $\varrho_1(a \to b) \simeq b$ for $a \to b \in \mathrm{Fun}(\Delta^1, \mathcal{D}(k)) \simeq \mathcal{D}(kA_2)$ with $a, b \in \mathcal{D}(k)$.

- the functor ϱ_2 is defined as the right adjoint of the tensor functor

$$(\text{-}) \otimes_k (k[-1] \to 0) \colon \mathcal{D}(k) \to \mathcal{D}(kA_2)\,.$$

  Explicitly, we have that $\varrho_2(a \xrightarrow{\alpha} b) \simeq \operatorname{cof}(\alpha)$.

- the functor $\varrho_3[-1]$ is defined as the right adjoint of the tensor functor

$$(\text{-}) \otimes_k (k \xrightarrow{\mathrm{id}} k) \colon \mathcal{D}(k) \to \mathcal{D}(kA_2)\,.$$

  Explicitly, we have that $\varrho_3[-1](a \to b) \simeq a$.

By cofinality [Lur09, §4.1], the ∞-category of global sections of $\mathcal{F}$, i.e. the limit of (6), is equivalent to the limit of the following sub-diagram:

$$\begin{tikzcd}
 & \mathcal{D}(k) & \\
\mathcal{D}(kA_2) \arrow[ur, "\varrho_1"] \arrow[dr, "\varrho_3[-1]"'] & & \mathcal{D}(kA_2) \arrow[ul, "\varrho_1"'] \arrow[dl, "\varrho_3[-1]"] \\
 & \mathcal{D}(k) &
\end{tikzcd}$$

This limit is in turn equivalent to the limit (pullback) in LinCat_k of the following diagram (this follows for instance from [Lur09, 4.2.3.10]).

$$\begin{tikzcd}
\mathcal{D}(kA_2) \arrow[dr, "{(\varrho_3[-1],\varrho_1)}"'] & & \mathcal{D}(kA_2) \arrow[dl, "{(\varrho_3[-1],\varrho_1)}"] \\
 & \mathcal{D}(k) \times \mathcal{D}(k) &
\end{tikzcd}$$

We can compute the above pullback as the pushout of the left adjoint diagram in $\mathcal{P}r^L_{\mathrm{St}}$, or equivalently in LinCat_k, since the functor $\mathrm{LinCat}_k \to \mathcal{P}^L_{\mathrm{St}}$ preserves colimits. This left adjoint diagram is equivalent to the image under the functor $\mathcal{D}(\text{-}) \colon \mathrm{dgCat}_k \to \mathrm{LinCat}_k$ of the following diagram of dg-categories.

$$\begin{tikzcd}
 & k \amalg k \arrow[dl, "\phi"'] \arrow[dr, "\phi"] & \\
kA_2' & & kA_2'
\end{tikzcd} \tag{7}$$

Here $k \amalg k$ refers to the dg-category with two objects $1, 2$, and having as morphisms, up to k-linear multiple, only identities or zero maps. The dg-category kA_2' is Morita equivalent to the A_2-quiver, it is given by the dg-category with two objects $1, 2$ and

$$\mathrm{Hom}_{kA_2'}(i,j) = \begin{cases} k & i = j \text{ or } i = 1, j = 2 \\ 0 & i = 2, j = 1 \end{cases} \in \mathrm{Ch}(k)\,.$$

Under the Morita equivalence $\mathrm{dgMod}^{\mathrm{perf}}(kA_2') \simeq \mathrm{dgMod}^{\mathrm{perf}}(kA_2)$, the two objects $1, 2 \in kA_2'$ are identified with the two projectove kA_2-modules $P_1 = (k \xrightarrow{\mathrm{id}} k)$ and $P_2 = (0 \to k)$. The morphism of dg-categories ϕ is the apparent cofibration mapping 1 to 1 and 2 to 2. Passing to the derived ∞-category, the inclusion ϕ of the objects $1, 2$ gives to the tensor functor with the projective module P_1, P_2, this is the right adjoint of (ϱ_1, ϱ_2).

Since the diagram (7) is cofibrant, its homotopy colimit coincides with the 1-categorical colimit, which is given by the dg-version of the Kronecker quiver $Q = \ 1 \rightrightarrows 2$. We thus find that

$$\mathcal{H}(\Gamma, \mathcal{F}) \simeq \mathcal{D}(Q)\,.$$

The ∞-category $\mathcal{H}(\Gamma, \mathcal{F})$ is also called the topological Fukaya category of the annulus. More generally, the topological Fukaya category of any framed marked surface [DK15, DK18] arises as the global sections of constructible sheaf on trivalent graph embedded in the surfaces, that assign $\mathcal{D}(kA_2)$ to each vertex and $\mathcal{D}(k)$ to each edge.

If $k = \mathbb{C}$, the ∞-category $\mathcal{H}(\Gamma, \mathcal{F})$ is furthermore equivalent to the derived ∞-category of quasi-coherent sheaves $\mathcal{D}(\mathbb{P}^1)$ on $\mathbb{P}^1$.

2.3 More on limits in ∞-categories of ∞-categories

In this subsection, we describe the computation of limits in Cat_∞ in terms of coCartesian sections of the Grothendieck construction. Since the forgetful functors in (5) all preserve limits, this tells one how to compute limits in a number of ∞-categories of ∞-categories.

This description in terms of sections of the Grothendieck construction formalizes a very simple insight: an object in the limit of a diagram, i.e. a functor from the point $* = \Delta^0$ to the limit, is by the universal property of the limit the same thing as a compatible family of objects in the values of the diagram. Stated differently, a global section of the constructible sheaf is given by a compatible family of local sections[4]. If we express a complicated stable ∞-category $\mathcal{C}$ as the limit of a diagram of simpler stable ∞-categories, we can thus (sometimes greatly) simplify the construction and study of the objects of $\mathcal{C}$.

Let Z be a 1-category and $f : Z \to \mathrm{Set}_\Delta$ a functor valued in ∞-categories. The functor f defines a functor $F : N(Z) \to \mathrm{Cat}_\infty$, which we can think of as a strictly commuting diagram of ∞-categories.

Definition 2.9 ([Lur09, 3.2.5.2])**.** We define the Grothendieck construction $p\colon \Gamma(f) \to N(Z)$ (referred to as the relative nerve in [Lur09]) as as the following coCartesian fibration between ∞-categories. Firstly, simplicial set $\Gamma(f)$ is defined, such that an n-simplex $\Delta^n \to \Gamma(f)$ amounts to:

- A functor $\sigma\colon \Delta^n \to N(Z)$.
- For every simplicial subset Δ^I, $I = \{i_1, \dots, i_m\} \subset [n]$, a functor $\Delta^I \to f(\sigma(i_m))$.
- For every pair of $I = \{i_1, \dots, i_m\} \subset J = \{j_1, \dots, j_{m'}\} \subset [n]$ the arising diagram

$$\begin{tikzcd} \Delta^I \arrow[r] \arrow[d, hook] & f(\sigma(i_m)) \arrow[d, "f\circ\sigma(i_m \to j_{m'})"] \\ \Delta^J \arrow[r] & f(\sigma(j_{m'})) \end{tikzcd}$$

  is required to commute.

The functor $p\colon \Gamma(f) \to N(Z)$ is defined by mapping an n-simplex as above to the n-simplex σ in $N(Z)$.

An object in $\Gamma(f)$ thus consists of a choice of $z \in N(Z)$ and an object $X_z \in f(z)$, we denote the object by (z, X_z). A morphism $(z, X_z) \to (z', X_{z'})$ in $\Gamma(f)$ consists of a morphism $\alpha\colon z \to z'$ together with a morphism $f(\alpha)(X_z) \to X_{z'}$ in $f(z')$.

[4]This universal property is one of the main motivations for us to formulate everything using sheaves and limits of ∞-categories, instead of using cosheaves and colimits of ∞-categories.

Definition 2.10.

(1) A section of the Grothendieck construction $p\colon \Gamma(f) \to N(Z)$ consists of a functor $X\colon N(Z) \to \Gamma(f)$ whose composite with p is the identity on $N(Z)$. We denote by

$$\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(f)) \coloneqq \mathrm{Fun}(N(Z), \Gamma(f)) \times_{\mathrm{Fun}(N(Z),N(Z))} \{\mathrm{id}_{N(Z)}\}$$

the ∞-category of sections of p.

(2) A section X of p is called coCartesian[5] if for each morphism $\alpha : i \to j$ in $N(Z)$, the morphism $X(\alpha)\colon (i, X(i)) \to (j, X(j))$ in $\Gamma(f)$, with $i, j \in Z$ and $X(i) \in f(i)$, $X(j) \in f(j)$, encodes an equivalence $f(\alpha)(X(i)) \simeq X(j)$. We denote by

$$\mathrm{Fun}^{\mathrm{coCart}}_{N(Z)}(N(Z), \Gamma(f)) \subset \mathrm{Fun}_{N(Z)}(N(Z), \Gamma(f))$$

the full subcategory of coCartesian sections.

Theorem 2.11. *The limit of $F\colon N(Z) \to \mathrm{Cat}_\infty$ is equivalent to $\mathrm{Fun}^{\mathrm{coCart}}_{N(Z)}(N(Z), \Gamma(f))$.*

Proof. This appears as Corollary 7.4.1.10 in Lurie's Kerodon [Lur]. A similar statement also appears in [Lur09, 3.3.3.2] (warning: there are some typos in that statement). □

Remark 2.12. One can perform actual down to earth computations using this description in terms of coCartesian sections of the Grothendieck construction. Further, there is the neat feature that one can decompose some computations into a collection of much simpler computations in the larger ∞-category of all sections of the Grothendieck construction. This allows for local-to-global arguments for, for instance, the computation of Homs between global sections. This circle of ideas was applied for instance in [Chr21, CHQ23].

Example 2.13. Consider the constructible sheaf from Example 2.8. The two modules over the Kronecker quiver

$$S_1 = \ k \rightrightarrows 0 \ , \qquad S_2 = \ 0 \rightrightarrows k$$

can be identified with the following coCartesian sections of the Grothendieck construction:

$$S_1 \simeq \quad \begin{tikzcd} & & 0 & \\ k[1] & (k \to 0) \arrow[l] \arrow[ur] \arrow[dr] & k[1] & (k \to 0) \arrow[l] \arrow[ul] \arrow[dl] \\ & & k & \end{tikzcd}$$

$$S_2 \simeq \quad \begin{tikzcd} & & k & \\ k & (0 \to k) \arrow[l] \arrow[ur] \arrow[dr] & k & (0 \to k) \arrow[l] \arrow[ul] \arrow[dl] \\ & & 0 & \end{tikzcd}$$

[5]One can check that such an $X(\alpha)$ defines a p-coCartesian edge in the sense of [Lur09, page 118 or the dual of Def. 2.4.1.1].

3 Applications to categorifications of cluster algebras

> It can be viewed as a manifestation of a general principle:
> All interesting categories are equivalent to Fukaya categories.
>
> A.B. Goncharov, *Ideal webs, moduli spaces of local systems, and 3d Calabi-Yau categories* [Gon17]

3.1 Marked surfaces and graphs

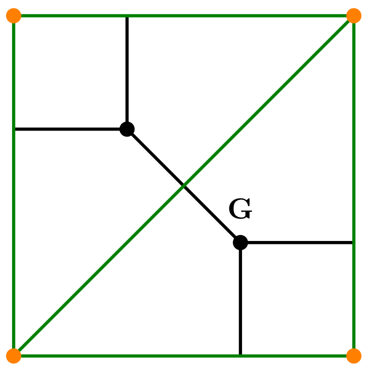
Definition 3.1. By a surface $\mathbf{S}$, we mean an oriented, compact, connected 2-dimensional topological manifold with nonempty boundary $\partial\mathbf{S}$. All connected boundary components of $\mathbf{S}$ are circles.

A marked surface $(\mathbf{S}, M)$, usually simply denoted by $\mathbf{S}$, consists of a surface with a finite nonempty subset $M \subset \partial\mathbf{S}$ of marked points on the boundary, satisfying that each boundary component contains a marked point.

Example 3.2. Two simple examples of marked surfaces: the 4-gon (disc with 4 boundary marked points) and the annulus with two marked points.

A triangulation of a marked surface $\mathbf{S}$ consists of a decomposition of $\mathbf{S}$ into triangles with corners at the marked points. The dual graph of a triangulation is obtained by placing a vertex into each triangle of the triangulation and connecting these vertices according to the incidence of the triangulation. Each boundary edge of the triangulation gives rise to an external edge of the graph. Note that this dual graph is trivalent.

Example 3.3. A triangulation of the 4-gon and its dual trivalent graph $\mathbf{G}$.



Remark 3.4. The (clockwise) orientation of $\mathbf{S}$ equips the dual graph with the structure of a ribbon graph, meaning a graph with a cyclic orientation of the halfedges incident to any given vertex.

3.2 Relative Ginzburg algebras of triangulated surfaces

Let $\mathbf{S}$ be a marked surface with a triangulation and dual trivalent ribbon graph $\mathbf{G}$. In the following, we define a constructible cosheaf $\mathcal{F}^{\mathrm{dg}}\colon \operatorname{Exit}(\mathbf{G})^{\mathrm{op}} \to \mathrm{dgCat}_k$ of dg-categories on $\mathbf{G}$.

Construction 3.5. Let v be a vertex of $\mathbf{G}$ with set E_v of incident edges. The set E_v has three objects and carries the clockwise cyclic order, we write $x+1$ for the successor. We define $\mathcal{F}^{\mathrm{dg}}(v)$ to be the dg-category

- with set of objects E_v
- morphisms freely generated by morphisms $a_x : x \to x+1$ in degree 0, morphisms $b_{x+2} : x \to x+2$ in degree 1, as well as morphisms $l_x, L_x\colon x \to x$ with $\deg(l_x) = 1, \deg(L_x) = 2$, where $x \in E_v$ is any object, and
- the differential determined on the free generators by

$$d(a_x) = 0$$

$$d(b_{x+2}) = a_{x+1}a_x$$

$$d(l_x) = 0$$

$$d(L_x) = l_x - a_{x-1}b_{x-1} + b_x a_x$$

Labeling the edge incident to v by $1, 2, 3$ (compatibly with their cyclic order), we can depict the objects and free generating morphisms of $\mathcal{F}^{\mathrm{dg}}(v)$ as follows:

$$\begin{tikzcd}
& 2 \arrow["l_2"', loop above] \arrow["L_2", out=135, in=45, loop, distance=3em] \arrow[dl, "b_1"'] \arrow[dr, bend left, "a_2"] & \\
1 \arrow[ur, bend left, "a_1"] \arrow[rr, "b_3"] \arrow["l_1", loop left] \arrow["L_1"', out=210, in=150, loop, distance=3em] & & 3 \arrow[ul, "b_2"'] \arrow[ll, bend left, "a_3"] \arrow["l_3"', loop right] \arrow["L_3", out=30, in=-30, loop, distance=3em]
\end{tikzcd}$$

Note that $\mathcal{F}^{\mathrm{dg}}(v)$ is quasi-isomorphic to the dg-category obtained by the same recipe but removing the six cycles l_i, L_i; the cycles are needed for the cofibrancy of the dg-functor below.

For e an edge of $\mathbf{G}$, we let $\mathcal{F}^{\mathrm{dg}}(e)$ be the dg-category with a single object e and morphisms freely generated by the endomorphism $l_e\colon e \to e$ lying in degree 1 with $d(l_e) = 0$.

For v a vertex of $\mathbf{G}$ and e an incident edge, we define the dg-functor $\mathcal{F}^{\mathrm{dg}}(e \to v)\colon \mathcal{F}^{\mathrm{dg}}(e) \to \mathcal{F}^{\mathrm{dg}}(v)$, by mapping the object e to e and the endomorphism l_e to l_e.

Exercise 5. Show that the dg-functor

$$\prod_{e \in E_v} \mathcal{F}^{\mathrm{dg}}(e \to v)\colon \prod_{e \in E_v} \mathcal{F}(e) \to \mathcal{F}(v)$$

is a cofibration in the quasi-equivalence model structure.

Lemma 3.6 ([CHQ23, Lemma 6.21])**.** *The diagram $\mathcal{F}^{\mathrm{dg}}$ is cofibrant (with respect to the projective model structure on* $\operatorname{Fun}(\operatorname{Exit}(\mathbf{G})^{\mathrm{op}}, \mathrm{dgCat}_k)$ *induced by the quasi-equivalence model structure on* dgCat_k*). In particular, the homotopy colimit of $\mathcal{F}^{\mathrm{dg}}$ coincides with the (1-categorical) colimit.*

Definition 3.7. The relative Ginzburg dg-category $\mathscr{G}$ of the triangulated marked surface $\mathbf{S}$ is defined as the colimit of $\mathcal{F}^{\mathrm{dg}}$ in dgCat_k.

We remark that the dg-category $\mathscr{G}$ is Morita equivalent to a dg-algebra referred to as the relative Ginzburg algebra associated with the triangulation, see for instance [Chr22b] for a definition.

Let $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{LinCat}_k$ be the constructible sheaf given by the right adjoint diagram of $\mathcal{D}(\text{-}) \circ \mathcal{F}^{\mathrm{dg}}$.

Theorem 3.8 ([Chr22b])**.** *There exists an equivalence between k-linear ∞-categories*

$$\mathcal{H}(\mathbf{G}, \mathcal{F}) \simeq \mathcal{D}(\mathscr{G})\,.$$

Proof. Combine Lemma 3.6 with the results for how to compute limits and colimits of stable ∞-categories, see Sections 1.5 and 2.2. □

The relative Ginzburg algebras described above are part of the more general class of relative Ginzburg algebras associated with ice quivers with potential, see [Wu23]. If the ice quiver has no frozen vertices, one speaks of a (non-relative) Ginzburg algebra. Such Ginzburg algebras and their relative versions are interesting for many reasons. Here are a few:

- They can be used for the construction of cluster categories, which categorify cluster algebras, as we will discuss further below.
- The derived category of finite dimensional modules over the non-relative Ginzburg algebra associated with a triangulated surface embeds fully faithfully into the derived Fukaya category of a Calabi–Yau threefold Y with a fibration $\pi\colon Y \to \mathbf{S}$ to the surface [Smi15]. It seems likely that the perfect derived category of $\mathscr{G}$ describes a partially wrapped Fukaya category of this threefold Y.
- Ginzburg algebras are under mild assumptions left 3-Calabi–Yau [Kel11] and relative Ginzburg algebras are under mild assumptions relative left 3-Calabi–Yau in the sense of Brav-Dyckerhoff [BD19], see [Wu23, Yeu16]. Further, under mild assumptions any (relative) left 3-Calabi–Yau dg-algebra is quasi-isomorphic to a (relative) Ginzburg algebra [dB15, KL23].
- The spaces of Bridgeland stability conditions of (relative) Ginzburg algebras coming from surfaces can be described in terms of spaces of quadratic differentials [BS15, KQ20, CHQ23].

3.3 The 1-periodic topological Fukaya category

We begin with what should initially be a surprising fact. Consider the derived ∞-category $\mathcal{F}(v) = \mathcal{D}(\mathcal{F}^{\mathrm{dg}}(v))$ of the dg-algebra associated with a vertex in the last section.

Lemma 3.9 ([Chr22b])**.** *The ∞-category $\mathcal{F}(v)$ is equivalent to the lax limit of the diagram in (the $(\infty,2)$-categorical version of)* LinCat_k

$$\mathcal{D}(k) \xrightarrow{\text{-}\otimes k} \mathcal{D}(k[t_1]) \xrightarrow{\mathrm{id}} \mathcal{D}(k[t_1])$$

with $k[t_1]$ the graded algebra of polynomials (considered as a dg-algebra) with generator in degree $|t_1| = 1$ and the first functor arises from the apparent k-$k[t_1]$-bimodule with underlying complex k. In particular, $\mathcal{F}(v)$ admits a semiorthogonal decomposition

$$(\mathcal{D}(k), \mathcal{D}(k[t_1]), \mathcal{D}(k[t_1]))\,.$$

Proof sketch. The (derived) endomorphism algebra of any object $x \in \mathcal{F}^{\mathrm{dg}}(v)$ in $\mathcal{F}(v)$ is equivalent to $k[t_1]$. Thus, the presentable subcategory of $\mathcal{F}(v)$ generated by x is equivalent to $\mathcal{D}(k[t_1])$ by [Lur17, 7.1.2.1]. Further, one can check that $\mathrm{Mor}_{\mathcal{F}(v)}(x, x+1) \simeq k[t_1]$ and $\mathrm{Mor}_{\mathcal{F}(v)}(x+1, x) \simeq 0$. This gives rise to the two copies of $\mathcal{D}(k[t_1])$ in the semiorthogonal decomposition. Since the composite morphism $x \xrightarrow{a_x} x+1 \xrightarrow{a_{x+1}} x+2$ vanishes (more precisely, we have a specific null homotopy in the dg-nerve arising from the identity $d(b_{x+2}) = a_{x+1}a_x$), we get a morphism $\mathrm{fib}(x \xrightarrow{a_x} x+1) \to x+2[1]$, whose fiber y satisfies $\mathrm{Mor}_{\mathcal{F}(v)}(y, y) \simeq k$, and thus generates the full subcategory $\mathcal{D}(k)$ inside of $\mathcal{F}(v)$. One computes that

$$\mathrm{Mor}_{\mathcal{F}(v)}(y, x+i) \simeq \begin{cases} k & i = 0 \\ 0 & i = 1 \end{cases}$$

and

$$\mathrm{Mor}_{\mathcal{F}(v)}(x+i, y) \simeq \begin{cases} 0 & i = 0 \\ 0 & i = 1 \end{cases}$$

From this, one deduces the desired semiorthogonal decomposition, as well as determines the gluing functors, leading to the description as the lax limit. □

Let $k[t_1^{\pm}]$ be the graded algebra of Laurent polynomials with generator in degree $|t_1| = 1$. Its derived ∞-category $\mathcal{D}(k[t_1^{\pm}])$ describes the derived ∞-category of 1-periodic chain complexes, meaning chain complexes $C_\bullet$ together with an identification $t_1 \colon C_\bullet \simeq C_{\bullet+1}$. We remark that $\mathcal{D}(k[t_1^{\pm}])$ is not 1-periodic in the sense that $[1] = \mathrm{id}_{\mathcal{D}(k[t_1^{\pm}])}$ (though this holds on objects) if $\mathrm{char}(k) \neq 2$. However $\mathcal{D}(k[t_1^{\pm}])$ is 2-periodic and can be considered as a $k[t_2^{\pm}]$-linear ∞-category.

Pulling back along the morphism of dg-algebras $k[t_1] \xrightarrow{t_1 \mapsto t_1} k[t_1^{\pm}]$ yields a functor $\mathcal{D}(k[t_1^{\pm}]) \to \mathcal{D}(k[t_1])$.

Exercise 6. Show that the functor $\mathcal{D}(k[t_1^{\pm}]) \to \mathcal{D}(k[t_1])$ is fully faithful.
Hint: show that $\mathrm{Mor}_{\mathcal{D}(k[t_1])}(k[t_1^{\pm}], k[t_1^{\pm}]) \simeq \mathrm{Mor}_{\mathcal{D}(k[t_1^{\pm}])}(k[t_1^{\pm}], k[t_1^{\pm}]) \simeq k[t_1^{\pm}]$.

Remark 3.10. The subcategory $\mathcal{D}(k[t_1^{\pm}]) \subset \mathcal{D}(k[t_1])$ describes the kernel of the right adjoint $\mathrm{Mor}_{\mathcal{D}(k[t_1])}(k, \text{-}) \colon \mathcal{D}(k[t_1]) \to \mathcal{D}(k)$ of $\text{-} \otimes k$.

Lemma 3.11 ([Chr22a])**.** *There is a constructible subsheaf $\mathcal{F}^{\mathrm{clst}} \subset \mathcal{F}$ with*

- *for any edge e of $\mathbf{G}$*
$$\mathcal{F}^{\mathrm{clst}}(e) = \mathcal{D}(k[t_1^{\pm}]) \subset \mathcal{D}(k[t_1]) = \mathcal{F}(e)$$
- *and for v any vertex of $\mathbf{G}$, the ∞-category $\mathcal{F}^{\mathrm{clst}}(v)$ given by the presentable subcategory generated from the two subcategories $\mathcal{D}(k[t_1^{\pm}]) \subset \mathcal{D}(k[t_1])$ in $\mathcal{F}(v)$. Thus*
$$\mathcal{F}^{\mathrm{clst}}(v) \simeq \mathrm{Fun}(\Delta^1, \mathcal{D}(k[t_1^{\pm}]))\,.$$

The three arising functors

$$\mathcal{F}^{\mathrm{clst}}(v) \simeq \mathrm{Fun}(\Delta^1, k[t_1^{\pm}]) \to \mathcal{D}(k[t_1^{\pm}]) = \mathcal{F}^{\mathrm{clst}}(e)$$

are given by the two evaluation functors at $0, 1$ and the cofiber functor (at least on objects).

Definition 3.12. The ∞-category of global sections $\mathcal{H}(\mathbf{G}, \mathcal{F}^{\mathrm{clst}})$ is called the 1-periodic topological Fukaya category of $\mathbf{S}$. We will write $\mathcal{C}_{\mathbf{S}} \coloneqq \mathcal{H}(\mathbf{G}, \mathcal{F}^{\mathrm{clst}})$.

Remark 3.13. The morphism of sheaves $\mathcal{F}^{\mathrm{clst}} \to \mathcal{F}$ induces a fully faithful functor

$$\mathcal{C}_{\mathbf{S}} = \mathcal{H}(\mathbf{G}, \mathcal{F}^{\mathrm{clst}}) \to \mathcal{H}(\mathbf{G}, \mathcal{F}) \simeq \mathcal{D}(\mathscr{G}) \,.$$

In fact, this functor has a left adjoint that exhibits $\mathcal{C}_{\mathbf{S}}$ as the Verdier quotient $\mathcal{D}(\mathscr{G}) / \operatorname{Ind} \mathcal{D}^{\mathrm{fin}}(\mathscr{G})$, where $\mathcal{D}^{\mathrm{fin}}(\mathscr{G})$ denotes the derived ∞-category of modules with finite dimensional homology.

Remark 3.14. The ∞-category $\mathcal{C}_{\mathbf{S}}$ does not depend on the choice of triangulation (and thus dual graph $\mathbf{G}$) of $\mathbf{S}$, nor any other choices made above in its construction, up to equivalence of ∞-categories. We won't be able to prove this here, but a very clean proof, constructing this sheaf from a cyclic 2-Segal object, is given in [DK18].

3.4 Geometric models

The universal property as a limit of the ∞-category of global sections of a constructible sheaf implies that a global sections amounts to a compatible family of local sections. In this section, we briefly sketch how this can be used to associate global sections to suitable curves in $\mathbf{S}$.

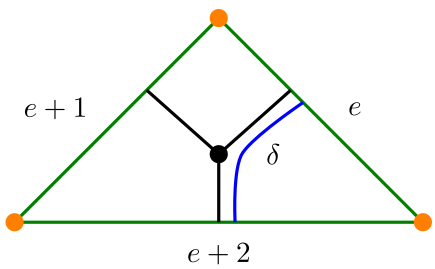
To begin with, let us look at some local sections of the perverse sheaf $\mathcal{F}$: given a vertex v of the trivalent graph $\mathbf{G}$, there are three objects in $\mathcal{F}^{\mathrm{dg}}(v)$, corresponding to the three edges of $\mathbf{G}$ incident to v, which give rise to three objects in $\mathcal{F}(v)$. For e such an edge, the functor $\mathcal{F}(v) \to \mathcal{F}(e)$ is given by $\operatorname{Mor}_{\mathcal{F}(v)}(e, \text{-})$, and as we noted in Section 3.3, we have

$$\operatorname{Mor}_{\mathcal{F}(v)}(e, e) \simeq \operatorname{Mor}_{\mathcal{F}(v)}(e+2, e) \simeq k[t_1] \in \mathcal{D}(k[t_1]) = \mathcal{F}(e) = \mathcal{F}(e+2)$$

and

$$\operatorname{Mor}_{\mathcal{F}(v)}(e+1, e) \simeq 0 \in \mathcal{D}(k[t_1]) = \mathcal{F}(e+1) \,.$$

We can think of the local section $e \in \mathcal{F}(v)$ as a curve δ in the 3-gon, with $\mathbf{G}$ in black:



The endpoints of the curve δ correspond to the edges where the local section $e \in \mathcal{F}(v)$ evaluates non-trivially. We call the curve δ a segment. Given any curve $\gamma\colon [0,1] \to \mathbf{S}$ which does not hit the vertices of $\mathbf{G}$, we can intersect γ with the triangulation to produce local segments of the curve. The local segments of the curve give rise to a compatible family of local sections, which glue to a global section, which we denote by $X_\gamma \in \mathcal{D}(\mathscr{G})$.

Many interesting objects in $\mathcal{D}(\mathscr{G})$ can be obtained from such gluing constructions, and these can be used to efficiently study their properties; this was explored in [Chr21]. Let us indicate a few examples of such results:

- Let $x \in \mathscr{G}$ be an object. There exists a curve γ as above and an equivalence $x \simeq X_\gamma$.
- Let $\gamma\colon [0,1] \to \mathbf{S}$ be an embedded curve disjoint to the vertices of $\mathbf{G}$. Then

$$\operatorname{Mor}_{\mathcal{D}(\mathscr{G})}(X_\gamma, X_\gamma) \simeq k[t_1] \,.$$

- More generally, given two suitable curves γ, γ', then $\mathrm{Mor}_{\mathcal{D}(\mathscr{G})}(X_\gamma, X_{\gamma'})$ is the direct sum of copies of shifts of $k[t_1]$, the number of which counts crossing of γ and γ' as well as so-called directed boundary intersections.

As shown in see [Chr22a], all indecomposable, compact objects in the subcategory $\mathcal{H}(\mathbf{G}, \mathcal{F}^{\mathrm{clst}}) \to \mathcal{H}(\mathbf{G}, \mathcal{F}) \simeq \mathcal{D}(\mathscr{G})$ can be associated with curves. We collect the results:

Definition 3.15. An allowed curve is a continuous map $\gamma\colon U \to \mathbf{S}\backslash M$ with $U = [0,1], S^1$, satisfying that

- all existent endpoints of γ (possibly none) lie in $\partial\mathbf{S}\backslash M$.
- away from the endpoints, γ is disjoint from $\partial\mathbf{S}$.
- γ does not cut out an unmarked disc in $\mathbf{S}$.

An open matching curve γ in $\mathbf{S}$ is an equivalence class of allowed curves under homotopies relative $\partial\mathbf{S}\backslash M$ with domain $U = [0,1]$ and considered up to reversal of orientation.

A closed matching curve is a suitable allowed curve with domain S^1 equipped with a local system of vector spaces, see [Chr22a].

Given a matching curve γ, we denote by $M_\gamma \in \mathcal{C}_{\mathbf{S}}$ the associated indecomposable object.

Theorem 3.16 (The geometrization Theorem)**.** *Let $X \in \mathcal{C}_{\mathbf{S}}$ be a compact object. Then there exists a unique and finite set J of matching curves in $\mathbf{S}$ and an equivalence in $\mathcal{C}_{\mathbf{S}}$*

$$X \simeq \bigoplus_{\gamma \in J} M_\gamma \,.$$

Let γ, γ' be two matching curves. We denote by $i^{\mathrm{cr}}(\gamma, \gamma')$ the number of crossings, meaning intersections in the interior (not counting redundant crossing which can be removed by changing the homotopy classes of the curves). We denote by $i^{\mathrm{bdry}}(\gamma, \gamma')$ the number of directed boundary intersections, meaning intersection with the same boundary component, such that γ' follows γ in the clockwise direction. This can be depicted as follows:

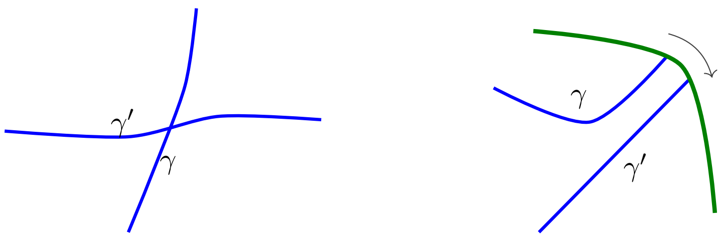


Figure 1: On the left: a crossing of two matching curves γ, γ'. On the right: a directed boundary intersection of two matching curves γ, γ'.

Theorem 3.17. *Let γ be an open matching curve. Then exists an equivalence*

$$\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_\gamma) \simeq k[t_1^{\pm}]^{\oplus 1 + 2 i^{\mathrm{cr}}(\gamma, \gamma')} \,.$$

Let γ, γ' be two distinct open matching curves. There exists an equivalence

$$\mathrm{Mor}_{\mathcal{C}_{\mathbf{S}}}(M_\gamma, M_{\gamma'}) \simeq k[t_1^{\pm}]^{\oplus i^{\mathrm{cr}}(\gamma, \gamma') \oplus i^{\mathrm{bdry}}(\gamma, \gamma')} \,.$$

3.5 The 2-Calabi–Yau exact ∞-structure

As we have seen in the last section, there are two types of morphisms between objects in $\mathcal{C}_{\mathbf{S}}$. Firstly, there are those coming from crossing intersections of matching curves, these morphisms are symmetric in the sense that the morphisms go in both directions. Secondly, there are those coming from directed boundary intersections, because of their directed nature the corresponding morphisms just go from one object to the other. We would like to think of the first class of morphisms as being Calabi–Yau, which we can formalize as follows:

Let e be an edge of $\mathbf{G}$. The limit cone of the diagram $\mathcal{F}^{\mathrm{clst}}$ contains a functor $\mathcal{C}_{\mathbf{S}} = \mathcal{H}(\mathbf{G}, \mathcal{F}^{\mathrm{clst}}) \to \mathcal{F}(e)$, which we denote by ev_e, we think of it as evaluating a global section at the edge e. Let $\mathbf{G}^{\partial}$ be the set of external edge of $\mathbf{G}$ and consider the product functor:

$$G \coloneqq \prod_{e\in \mathbf{G}^{\partial}} \mathrm{ev}_e \colon \prod_{e\in \mathbf{G}^{\partial}} \mathcal{H}(\mathbf{G}, \mathcal{F}^{\mathrm{clst}}) \longrightarrow \prod_{e\in \mathbf{G}^{\partial}} \mathcal{F}^{\mathrm{clst}}(e)$$

Definition 3.18. Let $X, Y \in \mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$. We denote by $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}}(X,Y) \subset \mathrm{Ext}^{1}_{\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}}(X,Y)$ the kernel of the morphism

$$G \colon \mathrm{Ext}^{1}_{\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}}(X,Y) \longrightarrow \mathrm{Ext}^{1}(G(X), G(Y))\,.$$

We call the k-vector space $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}^{\mathrm{c}}}(X,Y)$ the vector space of Calabi–Yau extensions; it forms a subfunctor of Ext^1.

An extension $M_\gamma \to M_{\gamma'}[1]$ corresponding to an intersection of γ and γ' is Calabi–Yau if and only if the intersection is a crossing and not a directed boundary intersection.

Proposition 3.19 ([Chr22a]). *There exists an equivalence of vector spaces*

$$\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(X,Y) \simeq \mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(Y,X)^{*}$$

bifunctorial in $X, Y \in \mathcal{C}_{\mathbf{S}}^{\mathrm{c}}$.

We can organize the Calabi–Yau extensions into an exact ∞-structure on $\mathcal{C}_{\mathbf{S}}$ in the sense of Barwick [Bar15]. Such an exact ∞-structure amounts to a collection of inflations $X \hookrightarrow Y$ and deflations $Y \twoheadrightarrow Z$ in $\mathcal{C}_{\mathbf{S}}$, satisfying ∞-categorical generalizations of the usual axioms of an exact 1-category. A fiber and cofiber sequence of the form $X \hookrightarrow Y \twoheadrightarrow Z$ is referred to as an exact sequence.

Lemma 3.20. *There exists the structure of an exact ∞-category on $\mathcal{C}_{\mathbf{S}}$, such that a fiber and cofiber sequence $X \to Y \to Z$ in $\mathcal{C}_{\mathbf{S}}$ is exact if and only if its image under G is a split fiber and cofiber sequence. The vector space of equivalence classes of exact extensions $X \to Y \to Z$ is in bijection with* $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}_{\mathbf{S}}}(Z,X)$.

This exact ∞-structure induces an extriangulated structure on the homotopy 1-category of $\mathcal{C}_{\mathbf{S}}$ in the sense of Nakaoka-Palu, see [NP20].

3.6 Cluster algebras of marked surface

We briefly recall the definition of cluster algebra, following the conventions of [FWZ16, Chapter 3]. Let $m_1, m_2 \geq 0$. Consider the field $\mathscr{F} = \mathbb{Q}(y_1, \ldots, y_{m_1+m_2})$ of rational functions in $m_1 + m_2$ variables.

Definition 3.21. A (labeled) seed $(\mathbf{x}, \tilde{M})$ in $\mathscr{F}$ consists of

- an $m_1 + m_2$-tuple $\mathbf{x} = (x_1, \ldots, x_{m_1+m_2})$ in $\mathscr{F}$ forming a free generating set of $\mathscr{F}$ and

- an $(m_1+m_2)\times m_1$-matrix $\tilde{M}$, such that the upper $m_1\times m_1$-matrix is skew-symmetric.

The tuple $\mathbf{x}$ is called a cluster and the elements $x_1,\dots,x_{m_1+m_2}$ are called cluster variables. The elements $x_{m_1+1},\dots,x_{m_1+m_2}$ are also called the frozen cluster variables. The matrix $\tilde{M}$ is called the extended mutation matrix.

Given such a seed $(\mathbf{x},\tilde{M})$ and a choice of $1\le l\le m_1$, we can produce a new seed $(\mathbf{x}',\mu_l(\tilde{M}))$ with

- $\mu_l(\tilde{M})_{i,j}=\begin{cases}-\tilde{M}_{i,j} & \text{if } i=l \text{ or } j=l\\ \tilde{M}_{i,j}+\tilde{M}_{i,l}\tilde{M}_{l,j} & \text{if } \tilde{M}_{i,l}>0 \text{ and } \tilde{M}_{l,j}>0\\ \tilde{M}_{i,j}-\tilde{M}_{i,l}\tilde{M}_{l,j} & \text{if } \tilde{M}_{i,l}<0 \text{ and } \tilde{M}_{l,j}<0\\ \tilde{M}_{i,j} & \text{else.}\end{cases}$
- $\mathbf{x}'=(x_1,\dots,x_{l-1},x_l',x_{l+1},\dots,x_{m_1+m_2})$, where x_l' is determined by the cluster exchange relation
$$x_l'x_l=\prod_{j \text{ with } \tilde{M}_{j,l}>0} x_j^{\tilde{M}_{j,l}}+\prod_{j \text{ with } \tilde{M}_{j,l}<0} x_j^{-\tilde{M}_{j,l}}.$$

The seed $(\mathbf{x}',\mu_l(\tilde{M}))$ is called the seed mutation of $(\mathbf{x},\tilde{M})$ at the cluster variable x_l.

Definition 3.22. Let $(\mathbf{x},\tilde{M})$ be a seed. The associated cluster algebra $\mathrm{CA}\subset\mathscr{F}$ is the $\mathbb{Q}$-subalgebra of $\mathscr{F}$ generated by all cluster variables in all seeds obtained from $(\mathbf{x},\tilde{M})$ via iterated seed mutation.

Given a marked surface $\mathbf{S}$, an arc in $\mathbf{S}$ is an embedded curve $\gamma\colon[0,1]\to\mathbf{S}$ with endpoints in M, and otherwise disjoint from M, considered up to homotopy and reversal of orientation. We forbid the (homotopy classes of) constant arcs. The edges of a triangulation of $\mathbf{S}$ can be considered as arcs. A boundary arc is an arc contained in $\partial\mathbf{S}$.

We now fix a marked surface $\mathbf{S}$ together with a maximal collection of non-crossing arcs, which thus decompose $\mathbf{S}$ into triangles. Such a collection of arcs is referred to as an (ideal) triangulation. Let m_1 be the number of interior arcs, labeled arbitrarily as $1,\dots,m_1$, and m_2 be the number of boundary arcs, labeled arbitrarily as $m_1+1,\dots,m_1+m_2$. For example, the triangulation in Example 3.3 has one interior arc and four boundary arcs.

Definition 3.23 ([FST08, Definition 4.1], [FT18])**.** The extended signed adjacency matrix $\tilde{M}=\sum_\Delta M_\Delta$ is the $(m_1+m_2)\times m_1$-matrix given by the sum over all ideal triangles Δ of the triangulation of the $(m_1+m_2)\times m_1$-matrices defined by

$$(M_\Delta)_{i,j}=\begin{cases}1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ with } i \text{ following } j \text{ in the counterclockwise direction,}\\ -1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ with } i \text{ following } j \text{ in the clockwise direction,}\\ 0 & \text{else.}\end{cases}$$

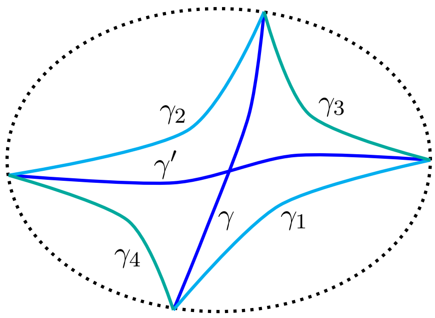
We obtain a labeled seed $(\mathbf{x},\tilde{M})$ with $\mathbf{x}=(y_1,\dots,y_{m_1+m_2})$ and denote by $\mathrm{CA}_{\mathbf{S}}$ the associated cluster algebra.

Theorem 3.24 ([FT18, Theorem 8.6])**.** *The cluster variables of $\mathrm{CA}_{\mathbf{S}}$ are canonically in bijection with the arcs in $\mathbf{S}$. The frozen cluster variables correspond to the boundary arcs. A set of cluster variables of $\mathrm{CA}_{\mathbf{S}}$ forms a cluster if and only if the corresponding arcs form an ideal triangulation of $\mathbf{S}$.*

Remark 3.25. The cluster exchange relations are given by the Ptolemy relations

$$\gamma\gamma'=\gamma_1\gamma_2+\gamma_3\gamma_4$$

for an arrangement of arcs as follows



where the arcs are understood to continue identically outside of the dotted circle.

3.7 The categorification of the cluster algebra

Definition 3.26. Let $T \in \mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$.

- We call T rigid (with respect to the exact structure on $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$) if $\mathrm{Ext}^{1,\mathrm{CY}}(T,T) \simeq 0$.
- We call T basic if T is the direct sum of non-equivalent (non-vanishing) indecomposable object.
- We call T maximal rigid if T is rigid, basic and maximal with this property, meaning that if $T \oplus Y$ is basic and rigid for some $Y \in \mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$, then $Y \simeq 0$.
- We call T cluster tilting if T is rigid, basic and $X \in \mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ is a direct sums of direct summands of T if and only if $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}}(T,X) \simeq 0$. Note that every cluster tilting object is maximal rigid.

Theorem 3.27 ([Chr22a])**.**

i) Let γ be a matching curve in $\mathbf{S}$. Then $M_\gamma \in \mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ is rigid if and only if the matching curve γ has no self-crossings. The collection of equivalence classes of rigid objects in $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ is thus in bijection with the collection of cluster variables of the cluster algebra $\mathrm{CA}_{\mathbf{S}}$.

ii) Consider a collection I of distinct matching curves in $\mathbf{S}$. Then $\bigoplus_{\gamma\in I} M_\gamma$ is a maximal rigid object if and only if it is a cluster tilting object, which is the case if and only if the matching curves in I have no crossings and decompose $\mathbf{S}$ into triangles. The collection of equivalence classes of cluster tilting objects in $\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}$ is thus in bijection with the collection of clusters of the cluster algebra $\mathrm{CA}_{\mathbf{S}}$.

Proof. Part i) follows from the fact that $\mathrm{Ext}^{1,\mathrm{CY}}_{\mathcal{C}^{\mathrm{c}}_{\mathbf{S}}}(M_\gamma, M_\gamma) \simeq k^{\oplus 2i^{\mathrm{cr}}(\gamma,\gamma)}$ vanishes if and only if γ has no self-crossings. Very similar arguments show that maximal rigid objects and cluster tilting objects coincide and are in bijection with ideal triangulations. □

The mutation of the clusters in the cluster algebra is categorified by a corresponding notion of mutation of cluster tilting objects. There is also a decategorification map, known as the cluster character, which allows to directly relate the objects in $\mathcal{C}_{\mathbf{S}}$ with the elements of the cluster algebra, see [Chr22a].

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