

∞ -categorical group quotients via skew group algebras

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Abstract

We study group quotients of dg categories and linear stable ∞ -categories. Given a group acting on a dg algebra, we prove that the skew group dg algebra is Morita equivalent to the dg categorical homotopy group quotient. We also treat the cases of group actions on dg categories, with corresponding skew group dg categories, and of orbit dg categories. Finally, we describe a version of the skew group algebra in the setting of ring spectra and relate it with ∞ -categorical group quotients.

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1 Introduction

Skew group algebras and skew group dg categories

The skew group algebra is a classical construction in the representation theory of finite dimensional algebras [RR85] and also appears in the construction of non-commutative resolutions of singularities [KV00, VdB22]. As input for its construction serves an algebra A over a commutative ring k and a group G acting on A by automorphisms. The skew group algebra AG has as underlying k -module the coproduct $A^{\amalg G}$, with multiplication determined on additive generators by $(g, a) \cdot (h, b) = (gh, a \cdot (g.b))$. An analogue of this construction is also known in the case that A is a dg algebra, dg category or even an A_∞ -category, see for instance [Meu20, OZ22, AP26].

The representation theory of the skew group algebra AG is typically better behaved than that of the algebra of fixed points A^G , which is why AG has become a standard construction. Indeed, under some finiteness assumptions, the module category $\text{mod}(AG)$ is equivalent to the category $\text{mod}(A)^G$ of G -equivariant A -modules, see [Dem11].

By passing to the setting of dg categories, we can clarify the universal property of the skew group algebra:

Theorem 1.1 (Theorem 3.5 and Remark 3.6). *Fix a base commutative ring k . Let A be a dg category with a strict action by a group G . We consider the G -action as a functor*

$$\rho: BG \rightarrow \mathrm{dgCat}_k,$$

where BG is the classifying space of G . The homotopy colimit¹ of ρ is Morita equivalent to the skew group dg category, denoted $A * G$.

If A has a single object, we can identify A with a dg algebra and in this case $A * G = AG$ describes the dg version of the skew group algebra.

The above homotopy colimit is also called the homotopy group quotient of A by G .

In the case $G = \mathbb{Z}$, we will also treat non-strict actions and show that the orbit dg category, recently constructed for non-strict $\mathbb{Z}$ -actions in [FKQ24], describes the homotopy colimit in Proposition 4.5.

To prove Theorem 1.1, we construct a strict cocone under a diagram equivalent to $\rho: BG \rightarrow \mathrm{dgCat}_k$, whose tip is Morita equivalent to $A * G$. Passing to derived ∞ -categories, we obtain an induced functor from the ∞ -categorical colimit $\mathcal{D}(A)_G$ to the derived ∞ -category of $A * G$. This functor $\mathcal{D}(A)_G \rightarrow \mathcal{D}(A * G)$ is an equivalence, which is shown by comparing the derived endomorphisms of generators. We use the ∞ -categorical colimit $\mathcal{D}(A)_G$ to describe the homotopy colimit, via the equivalence of dg categories up to Morita equivalence with compactly generated k -linear stable ∞ -categories.

The results of this paper unify different perspectives on categorical group quotients, leading to new results about these. For instance, exhibiting ∞ -categorical universal properties of the dg categorical constructions makes these accessible to powerful ∞ -categorical arguments. An immediate consequence of the above results is that the passage to skew group dg algebras and skew group dg categories commutes with homotopy colimits of dg categories, as colimits commute with colimits. Reversing the perspective, the results of this note also yield concrete models for the abstract ∞ -categorical constructions. We illustrate this by describing periodic derived ∞ -categories and periodic topological Fukaya categories of surfaces in terms of orbit categories in Section 4.3. Finally, we also obtain new results in the classical setting on group quotients of module 1-categories, see below.

Group actions on linear stable ∞ -categories

Let G be a group. The classifying space BG is the category with a single object $*$ with endomorphisms G . We work over a base $\mathbb{E}_\infty$ -ring spectrum R and denote by Mod_R the symmetric monoidal stable ∞ -category of R -module spectra. We denote by

$$\mathrm{LinCat}_{\mathrm{Mod}_R} = \mathrm{LMod}_{\mathrm{Mod}_R}(\mathcal{P}r^L)$$

the ∞ -category of Mod_R -linear presentable ∞ -categories (these are automatically stable).

A standard way to define an action of G on a Mod_R -linear ∞ -category $\mathcal{C}$ is as a functor $BG \rightarrow \mathrm{LinCat}_{\mathrm{Mod}_R}$, mapping $*$ to $\mathcal{C}$. Central to our treatment of group actions will be the following equivalent way to encode such G -actions, see for instance [CCRY25, BMCSY25]: forming the coproduct of Mod_R over G yields a monoidal ∞ -category $\mathrm{Mod}_R^{\amalg G} \in \mathrm{LinCat}_{\mathrm{Mod}_R}$, called the categorical group algebra; a G -action on $\mathcal{C}$ can equivalently be expressed as a left $\mathrm{Mod}_R^{\amalg G}$ -action on $\mathcal{C}$.

This perspective allows us to apply the powerful and well developed framework of ∞ -categorical algebra objects and modules of [Lur17] to the study of group actions and group quotients. For instance, the ∞ -categorical colimit over BG amounts in terms of left actions to tensoring with the Mod_R - $\mathrm{Mod}_R^{\amalg G}$ -bimodule Mod_R . Using this perspective, desired basic properties of group quotients readily follow.

Skew group ring spectra

In Section 5, we generalize the construction of the skew group algebra to the setting of a group G acting on an R -linear ring spectrum A . The skew group algebra AG is then an R -linear ring spectrum with underlying spectrum the coproduct $A^{\amalg G}$, and the multiplication

¹With respect to the Morita model structure on the category dgCat_k of dg categories.

generalizing the above. The actual construction of the ring spectrum AG is based on universal constructions, see Remark 5.4 for a summary. We show the following variant of Theorem 1.1 in this setting:

Theorem 1.2. *Let R be the base $\mathbb{E}_\infty$ -ring spectrum and $A \in \text{Alg}(\text{Mod}_R)$ be an R -linear ring spectrum. Let $\rho: BG \rightarrow \text{Alg}(\text{Mod}_R)$ be an action of a group G on A . The colimit of the functor between ∞ -categories*

$$BG \xrightarrow{\rho} \text{Alg}(\text{Mod}_R) \xrightarrow{\text{RMod}(\cdot)} \text{LinCat}_{\text{Mod}_R} \quad (1)$$

is equivalent to RMod_{AG} .

We remark that the limit of the functor (1) is equivalent to its colimit, since the theorem is formulated in the setting of presentable ∞ -categories, and hence to RMod_{AG} , see Lemma 2.7.

Comparison with previous results

Classical skew group algebras

Given a category C with an action by a group G , we denote by C^G the category of G -equivariant objects. Note that C^G describes the homotopy limit of a functor $BG \rightarrow \text{Cat}$, mapping the basepoint of BG to C , see Remark 3.8. Given a (not necessarily finite dimensional) algebra A over a commutative ring k , we denote by $\text{Mod}(A)$ the (non-small) abelian category of right A -modules. Using Theorem 1.1 and Lemma 2.7, we obtain the following:

Corollary 1.3. *Let k be a commutative ring and G a group acting on a k -algebra A . There exists an equivalence of derived ∞ -categories*

$$\mathcal{D}(A * G) \simeq \mathcal{D}(A)^G := \lim_{BG} \mathcal{D}(A).$$

Passing to homotopy 1-categories, this equivalence restricts to an equivalence of abelian subcategories $\text{Mod}(A * G) \simeq \text{Mod}(A)^G$.

If furthermore G is finite, k a field, and A finite dimensional over k , then $A * G$ is finite dimensional over k and this equivalence restricts to the abelian categories of finite dimensional modules: $\text{mod}(A * G) \simeq \text{mod}(A)^G$.

The equivalence of the small module categories from Corollary 1.3 was shown in the case that the characteristic of k does not divide the order of G in [Dem11, Prop. 2.48], but appears to be new in this generality. Corresponding equivalences of the bounded derived categories were deduced in [Ela14, Thm. 7.1] or [Che15, Bal11].

The proof of Corollary 1.3 is given in Section 3.

Ring spectra

Previous results in the literature concern a version of Theorem 1.2 in the case that $A = R$ is the base ring spectrum and G is a monoid in the ∞ -category of spaces (generalizing groups) acting on Mod_R . The corresponding results can be found in [Dou05, Prop. 3.13] or [HM25, Thm. 0.0.7] in case of $R = \mathbb{S}$ the sphere spectrum, and [CCRY25, Thm. 7.13] for general R . The analogue of the skew group algebra in this setting is called the Thom spectrum. The restriction $A = R$ corresponds to the assertion that the functor $BG \rightarrow \text{LinCat}_{\text{Mod}_R}$ is pointed. Note that in the case $A = R$, we also only allow the trivial G -action on the R -algebra $A = R$, since the G -action is assumed to be an R -linear algebra action.

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Higher categorical preliminaries

We generally follow the notation and conventions of [Lur09, Lur17].

- We consider dg categories over a commutative ring k . The derived ∞ -category $\mathcal{D}(\mathcal{B})$ of a dg category $\mathcal{B}$ is defined as the Ind-completion of the dg nerve $N_{\text{dg}}(\text{Perf}(\mathcal{B}))$ of the dg category of cofibrant compact right dg $\mathcal{B}$ -modules. The passage to the derived ∞ -category defines a colimit preserving and reflecting functor between ∞ -categories

$$\mathcal{D}(-): N(\text{dgCat}_k)[M^{-1}] \rightarrow \text{LinCat}_{\text{Mod}_k},$$

with M the collection of Morita equivalences, see also Remark 3.6.

- For a discussion of ∞ -operads, algebra objects and modules, we refer to Chapters 2 and 4 of [Lur17].
- R will usually denote a base $\mathbb{E}_\infty$ -ring spectrum. Given a Mod_R -linear ∞ -category $\mathcal{D}$ and $X, Y \in \mathcal{D}$, we write $\text{Mor}_{\mathcal{D}}(X, Y) \in \text{Mod}_R$ for the Mod_R -linear morphism object, see [Lur17, Def. 4.2.1.28].
- Given a regular cardinal κ , we denote by $\text{LinCat}_{\text{Mod}_R}^\kappa = \text{LMod}_{\text{Mod}_R}(\mathcal{P}r_\kappa^L)$ the presentable ∞ -category of Mod_R -modules in the presentable ∞ -category $\mathcal{P}r_\kappa^L$ of κ -compactly generated presentable ∞ -categories, as in [Lur17, Notation 5.3.2.8]. Its presentability is the advantage of $\text{LinCat}_{\text{Mod}_R}^\kappa$ over $\text{LinCat}_{\text{Mod}_R}$.

Given a Mod_R -linear monoidal ∞ -category $\mathcal{C} \in \text{Alg}(\text{LinCat}_{\text{Mod}_R})$, we choose a sufficiently large regular cardinal κ as in [Lur17, Lem. 5.3.2.12], for which in particular $\mathcal{C}$ is κ -compactly generated. We also denote

$$\text{LinCat}_{\mathcal{C}}^\kappa = \text{LMod}_{\mathcal{C}}(\text{LinCat}_{\text{Mod}_R}^\kappa)$$

for the presentable ∞ -category of κ -compactly generated $\mathcal{C}$ -linear categories.

- Given a presentable monoidal ∞ -category $\mathcal{C}$, in Sections 4.8.3-4.8.5 of [Lur17], Lurie describes a fully faithful (see [Lur17, Thm. 4.8.5.5]) functor²

$$\Theta_*: \text{Alg}(\mathcal{C}) \longrightarrow (\text{LinCat}_{\mathcal{C}})_{\mathcal{C}/},$$

mapping an algebra object $A \in \mathcal{C}$ to its ∞ -category $\text{RMod}_A(\mathcal{C})$ of right modules, together with the functor $- \otimes A: \mathcal{C} \rightarrow \text{RMod}_A(\mathcal{C})$ mapping the monoidal unit to A .

The image of the functor Θ_* is concretely characterized in [Lur17, Prop. 4.8.5.8]. The functor Θ is defined as the composite of Θ_* with the forgetful functor

$$(\text{LinCat}_{\text{Mod}_R})_{\text{Mod}_R/} \rightarrow \text{LinCat}_{\text{Mod}_R}.$$

2 ∞ -categorical group actions

For the entire section, we fix a group G (which is not required to be finite).

²More precisely, the functor Θ_* considered here is a restriction of the functor Θ_* of [Lur17], having additional functoriality in $\mathcal{C}$. We leave the choice of $\mathcal{C}$ in the notation for Θ_* implicit.

2.1 The categorical group algebra

Given a set (or a space) X , we denote by $\text{Mod}_R^{\coprod X}$ the colimit over X of the constant diagram in $\text{LinCat}_{\text{Mod}_R}$ with value Mod_R . Note that $\text{Mod}_R^{\coprod X} \simeq \text{Mod}_R^{\times X}$ also describes the limit over X , which follows from the equivalence $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\text{op}}$, see [Lur24, Cor. 06Q9], and the fact that the forgetful functors $\mathcal{P}r^R, \mathcal{P}r^L \rightarrow \text{Cat}_\infty$ preserve and reflect small limits, see [Lur24, Prop. 06PH].

There is an essentially unique colimit preserving functor

$$\text{Mod}_R^{\coprod(-)}: \mathcal{S} \rightarrow \text{LinCat}_{\text{Mod}_R} = \text{LMod}_{\text{Mod}_R}(\mathcal{P}r^L),$$

determined by mapping $*$ to Mod_R . Furthermore, since $\mathcal{S}$ is the unit object in the symmetric monoidal ∞ -category $\mathcal{P}r^L$, the functor $\text{Mod}_R^{\coprod(-)}$ lifts essentially uniquely to a symmetric monoidal functor $(\text{Mod}_R^{\coprod(-)})^\otimes: \mathcal{S}^\otimes \rightarrow \text{LinCat}_{\text{Mod}_R}^\otimes$, see [Lur17, Prop. 3.2.1.8].

Construction 2.1. The group G gives rise to an Assoc-algebra object in the Cartesian symmetric monoidal ∞ -category $\text{Set}^\otimes$ of sets, i.e. a morphism of ∞ -operads $\text{Assoc}^\otimes \rightarrow \text{Set}^\otimes$ over $N(\text{Fin}_*)$, see also [Lur17, Def. 4.1.1.3] for the definition of $\text{Assoc}^\otimes$. Note that such an Assoc-algebra object in $\text{Set}^\otimes$ simply amounts to a monoid.

Composing with the inclusion of symmetric monoidal ∞ -categories $\text{Set}^\otimes \subset \mathcal{S}^\otimes$ and the symmetric monoidal functor $(\text{Mod}_R^{\coprod(-)})^\otimes$, we obtain an Assoc-algebra object $\text{Mod}_R^{\coprod G} \in \text{Alg}_{\text{Assoc}}(\text{LinCat}_{\text{Mod}_R})$, meaning a Mod_R -linear monoidal ∞ -category.

We note that any group homomorphism $G \rightarrow G'$ induces a monoidal functor $\text{Mod}_R^{\coprod G} \rightarrow \text{Mod}_R^{\coprod G'}$.

Definition 2.2. The monoidal ∞ -category $\text{Mod}_R^{\coprod G}$ from Construction 2.1 is called the categorical group algebra of G .

Given $g \in G$, we denote the object of $\text{Mod}_R^{\coprod G}$ whose g -component has value R and whose other components vanish by R_g . The monoidal unit of $\text{Mod}_R^{\coprod G}$ is given by R_e .

Remark 2.3. The unique group homomorphism $\psi: G \rightarrow \{e\}$ gives by Construction 2.1 rise to a monoidal functor $\psi: \text{Mod}_R^{\coprod G} \rightarrow \text{Mod}_R$. Explicitly, ψ maps a G -tuple $(A_g)_{g \in G}$ to $\coprod_{g \in G} A_g$.

Lemma 2.4.

- (1) The monoidal ∞ -category $\text{Mod}_R^{\coprod G}$ is locally rigid in the sense of [Lur18, Def. D.7.4.1].
- (2) If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a $\text{Mod}_R^{\coprod G}$ -linear functor whose right adjoint F^R preserves colimits, then F^R inherits a $\text{Mod}_R^{\coprod G}$ -linear structure.

Proof. We begin with showing the local rigidity. Conditions (1) and (2) of Definition D.7.4.1 are clear. The unit object of $\text{Mod}_R^{\coprod G}$ is given by the object R_e , with e the unit of G , and clearly compact, giving condition (3).

The subset of left and right dualizable objects is closed under finite limits and colimits. For condition (4), it thus suffices to check that each object R_g with $g \in G$ admits a left and right dual. Since R_g is invertible with inverse $R_{g^{-1}}$, this is clear.

Part (2) of the Lemma follows from part (1) by [Lur18, Rem. D.7.4.4]. The broken reference at the end of loc. cit. may refer to the fact that the $\text{Mod}_R^{\coprod G}$ -linear structure on F^R in question is induced by the $\text{Mod}_R^{\coprod G}$ -linear structure of F via [Lur17, Cor. 7.3.2.12], so that the existence of the $\text{Mod}_R^{\coprod G}$ -linear structure in question is a property. $\square$

Example 2.5. We can consider the monoidal functor $\psi: \text{Mod}_R^{\coprod G} \rightarrow \text{Mod}_R$ as a $\text{Mod}_R^{\coprod G}$ -linear functor. Its right adjoint $\psi^R: \text{Mod}_R \rightarrow \text{Mod}_R^{\coprod G}$, G -componentwise given by the identity functor, inherits by Lemma 2.4 the structure of a $\text{Mod}_R^{\coprod G}$ -linear functor.

The composite $\psi^R \circ \psi: \text{Mod}_R^{\coprod G} \rightarrow \text{Mod}_R^{\coprod G}$ is equivalent to $\coprod_{g \in G} R_g \otimes (-)$ and inherits the structure of a monad.

The right adjoint $\psi^{RR}: \text{Mod}_R^{\coprod G} \rightarrow \text{Mod}_R$ of ψ^R is given by the product functor, as opposed to the coproduct functor ψ . Thus, if G is a finite group, then ψ^{RR} agrees with ψ . The inherited $\text{Mod}_R^{\coprod G}$ -linear structure of ψ^{RR} also coincides with that of ψ . This follows

from the fact that $\text{Mod}_R^{\text{II}G}$ is the free $\text{Mod}_R^{\text{II}G}$ -linear ∞ -category generated by a point, so that both linear functors are determined by the value of the monoidal unit R_e .

2.2 Group actions and group quotients

We denote by BG the classifying space of G , meaning the 1-category with a unique object $*$ and endomorphisms G . We will not distinguish between BG and its nerve $N(BG) \in \text{Set}_\Delta$ in notation.

A G -action on an object $\mathcal{C}$ in an ∞ -category $\mathcal{C}$ is understood to be a functor $BG \rightarrow \mathcal{C}$, mapping $*$ to $\mathcal{C}$.

Definition 2.6. Let $\mathcal{C}$ be an ∞ -category that admits limits and colimits. Given a G -action $\rho: BG \rightarrow \mathcal{C}$ on $\mathcal{C} \in \mathcal{C}$, we call

- the limit $\mathcal{C}^G := \lim(\rho)$ the fixed points of the G -action on $\mathcal{C}$.
- the colimit $\mathcal{C}_G := \text{colim}(\rho)$ the group quotient of $\mathcal{C}$ by G .

Lemma 2.7. Consider a functor $\rho_{\mathcal{C}}: BG \rightarrow \text{LinCat}_{\text{Mod}_R}$ describing a G -action on a Mod_R -linear ∞ -category $\mathcal{C}$. Its colimit $\mathcal{C}_G$ is equivalent to its limit $\mathcal{C}^G$ in $\text{LinCat}_{\text{Mod}_R}$.

Proof. The conservative functor $\text{LMod}_{\text{Mod}_R}(\mathcal{P}r^L) \rightarrow \mathcal{P}r^L$ reflects limits and colimits and preserves both by [Lur17, Cor. 3.4.3.6, 3.4.4.6]. The right adjoint diagram of the colimit diagram of $\rho_{\mathcal{C}}$ again lies in $\text{LMod}_{\text{Mod}_R}(\mathcal{P}r^L)$ by Lemma 2.4. It is a limit diagram, since its underlying diagram in $\mathcal{P}r^L$ is, as follows from the duality $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\text{op}}$ and the fact that the functors $\mathcal{P}r^R, \mathcal{P}r^L \rightarrow \text{Cat}_\infty$ preserve and reflect limits. Note that $BG^{\text{op}} \simeq BG$, since BG is a space, hence the right adjoint diagram of $\rho_{\mathcal{C}}$ is equivalent to $\rho_{\mathcal{C}}$. $\square$

We choose a sufficiently large regular cardinal κ . Then G -actions on κ -compactly generated Mod_R -linear ∞ -categories can be equivalently expressed as follows:

Proposition 2.8 ([CCRY25, BMCSY25]). *There exists a canonical equivalence of ∞ -categories*

$$\text{Fun}(BG, \text{LinCat}_{\text{Mod}_R}^\kappa) \simeq \text{LinCat}_{\text{Mod}_R^{\text{II}G}}^\kappa,$$

natural in the group G ³.

Proof. This is [CCRY25, Lem. 4.49] applied in the case $\mathcal{C} = \text{LinCat}_{\text{Mod}_R}^\kappa$ and $A = BG$. $\square$

We note that the naturality of the equivalence from Proposition 2.8 applied to the group homomorphism $*$ $\rightarrow$ G implies that the equivalence commutes with the forgetful functors to $\text{LinCat}_{\text{Mod}_R}^\kappa$.

Remark 2.9. The relation between a G -action $\rho: BG \rightarrow \text{LinCat}_{\text{Mod}_R}^\kappa$ and the $\text{Mod}_R^{\text{II}G}$ -linear structure on a given Mod_R -linear ∞ -category $\mathcal{C}$ can be explicitly described as follows: given $g \in G$, its action $\rho(g): \mathcal{C} \simeq \mathcal{C}$ is equivalent to the functor $R_g \otimes -: \mathcal{C} \simeq \mathcal{C}$.

Lemma 2.10 ([CCRY25, Lem. 4.51]). *Consider a group action $BG \rightarrow \text{LinCat}_{\text{Mod}_R}$ on $\mathcal{C}$ and the corresponding $\text{Mod}_R^{\text{II}G}$ -linear structure of $\mathcal{C}$ of Proposition 2.8.*

(1) *The group quotient of $\mathcal{C}$ is equivalent to the relative tensor product:*

$$\mathcal{C}_G \simeq \text{Mod}_R \otimes_{\text{Mod}_R^{\text{II}G}} \mathcal{C} \in \text{LinCat}_{\text{Mod}_R}. \quad (2)$$

(2) *The functor $\mathcal{C} \rightarrow \mathcal{C}_G$ appearing in the colimit cocone is equivalent to $\psi \otimes_{\text{Mod}_R^{\text{II}G}} \mathcal{C}$, with ψ as in Remark 2.3.*

³The functoriality on the left arises from left Kan extension and on the right by the relative tensor product.

Proof. The $\text{Mod}_R^{\text{H}G}$ -linear ∞ -category $\text{Mod}_R^{\text{H}G}$ inherits a G -action by Proposition 2.8. Using the naturality of the equivalence from Proposition 2.8 in the morphism $G \rightarrow *$, we find that

$$\text{colim}_{BG} \text{Mod}_R^{\text{H}G} \simeq \text{Mod}_R^{\text{H}G} \otimes_{\text{Mod}_R^{\text{H}G}} \text{Mod}_R \simeq \text{Mod}_R .$$

Using that the relative tensor product preserves colimits in the first entry, we find that

$$\mathcal{C}_G \simeq \text{colim}_{BG} (\text{Mod}_R^{\text{H}G} \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C}) \simeq \text{Mod}_R \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C} ,$$

showing statement (1). Statement (2) amounts to the fact that $\psi: \text{Mod}_R^{\text{H}G} \rightarrow \text{Mod}_R$ is the functor from the colimit cocone of $\text{Mod}_R^{\text{H}G}$.

The statements is also noted in [CCRY25, Lem. 4.51], using the fact that the inclusion $\text{LinCat}_{\text{Mod}_R}^{\kappa} \subset \text{LinCat}_{\text{Mod}_R}$ preserves colimits [Lur17, Lem. 5.3.2.9]. $\square$

Lemma 2.11. *Consider a group action $\rho_{\mathcal{C}}: BG \rightarrow \text{LinCat}_{\text{Mod}_R}$ on $\mathcal{C}$ and the corresponding $\text{Mod}_R^{\text{H}G}$ -linear structure.*

(1) *The functor*

$$\psi^R \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C}: \mathcal{C}_G \simeq \text{Mod}_R \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C} \longrightarrow \text{Mod}_R^{\text{H}G} \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C} \simeq \mathcal{C} \quad (3)$$

is monadic.

(2) *The endofunctor of $\mathcal{C}$ underlying the adjunction monad $\psi^R \psi \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C}$ of the adjunction $\psi \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C} \dashv \psi^R \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C}$ is given by $\coprod_{g \in G} \rho_{\mathcal{C}}(g)$.*

(3) *If G is finite, then the functor (3) is also left adjoint to $\mathcal{C} \otimes_{\text{Mod}_R^{\text{H}G}} \psi: \mathcal{C} \rightarrow \mathcal{C}_G$.*

Proof. We begin with showing (1). By Lemma 2.7, $\mathcal{C}_G$ is equivalent to the limit of $\rho_{\mathcal{C}}$, and hence to the ∞ -category of coCartesian sections of the Grothendieck construction of $\rho_{\mathcal{C}}$, see [Lur24, Prop. 05RX]. Under this equivalence, the functor

$$\mathcal{C} \otimes_{\text{Mod}_R^{\text{H}G}} \psi^R: \mathcal{C}_G \simeq \text{colim}(\rho_{\mathcal{C}}) \longrightarrow \mathcal{C}$$

evaluates a coCartesian section at $* \in BG$. Thus, the functor is clearly conservative and hence monadic by [Lur17, Thm. 4.7.3.5].

For (2), we note the equivalence

$$\psi^R \psi \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C} \simeq \prod_{g \in G} (R_g \otimes (-)) \otimes_{\text{Mod}_R^{\text{H}G}} \mathcal{C} \simeq \prod_{g \in G} \rho_{\mathcal{C}}(g) . \quad (4)$$

For (3), we note that by Example 2.5, if G is finite, the $\text{Mod}_R^{\text{H}G}$ -linear functor ψ^R is also left adjoint to ψ , which induces the desired adjunction. $\square$

3 Skew group dg categories

In this section, we consider a dg category A with a strict action by a group G . We relate the skew group dg category with the ∞ -categorical group quotient.

Recall that k is a choice of base commutative ring. We fix a group G with a strict action on a dg category A . We can equivalently consider the action as a functor $\rho_A: BG \rightarrow \text{dgCat}_k$ mapping $*$ to A . We define the skew group dg category $A * G$ as follows, analogous to the definition of the skew group A_{∞} -category in [OZ22, Def. 5.6], [AP26, Def. 2.11].

- The set of objects of $A * G$ is given by the set of objects of A . We will however write $\tilde{X} \in A * G$ for the object corresponding to $X \in A$.
- Given $\tilde{X}, \tilde{Y} \in A * G$, the morphism chain complex is given by

$$\text{Map}_{A * G}(\tilde{X}, \tilde{Y}) = \coprod_{g \in G} \text{Map}_A(g.X, Y) .$$

Given a morphism $a: g.X \rightarrow Y$ in A , we write $(g, a): \tilde{X} \rightarrow \tilde{Y}$ for the corresponding morphism in $A * G$. Every morphism in $A * G$ can be uniquely written as a sum $\sum_{g \in G} (g, a_g)$, with finitely many non-zero summands.

- The composition map

$$(-) \circ (-): \text{Map}_{A * G}(\tilde{Y}, \tilde{Z}) \times \text{Map}_{A * G}(\tilde{X}, \tilde{Y}) \longrightarrow \text{Map}_{A * G}(\tilde{X}, \tilde{Z})$$

is defined on generators by $(g_2, b) \circ (g_1, a) := (g_2 g_1, b \circ (g_2 \cdot a))$.

There is an apparent dg functor $F_A: A \rightarrow A * G$, given by the assignments $X \mapsto \tilde{X}$ on objects and $a \mapsto (e, a)$ on morphisms, where $e \in G$ is the unit element.

We can define a G -action $\rho_{A * G}: BG \rightarrow \text{dgCat}_k$ on $A * G$ by letting $h \in G$ act as $h \cdot \tilde{X} = \widetilde{h \cdot X}$ and $h \cdot (g, a) = (hgh^{-1}, h \cdot a)$. Indeed, $e \in G$ clearly acts as the identity, and we have

$$\begin{aligned} h \cdot ((g_2, b) \circ (g_1, a)) &= (hg_2 g_1 h^{-1}, h \cdot (b \circ (g_2 \cdot a))) \\ &= (hg_2 h^{-1}, h \cdot b) \circ (hg_1 h^{-1}, h \cdot a) = h \cdot (g_2, b) \circ h \cdot (g_1, a). \end{aligned}$$

The dg functor $A \rightarrow A * G$ is clearly G -equivariant, which we can express as follows:

Lemma 3.1. *The dg functor $F_A: A \rightarrow A * G$ extends to a morphism $\rho_A \rightarrow \rho_{A * G}$ in $\text{Fun}(BG, \text{dgCat}_k)$.*

Definition 3.2. We choose for every G -orbit $[X]$ of objects in A an arbitrary representative $X \in A$. The dg category $(A * G)^{\text{red}}$ is defined as the full dg subcategory of $A * G$ on the objects of the form $\tilde{X}$, with X a chosen representative.

We note that the inclusion $(A * G)^{\text{red}} \subset A * G$ is an equivalence of dg categories.

Lemma 3.3. *Suppose that the action of G is free on the set of objects of A . Then there exists an equivalence of dg categories $A * G \rightarrow (A * G)^{\text{red}}$ that is (strictly) G -equivariant with respect to the trivial G -action on $(A * G)^{\text{red}}$.*

Proof. There is an apparent inverse E to the equivalence $i: (A * G)^{\text{red}} \subset A * G$, mapping each object $\tilde{X} \in A * G$ to the representative $E(\tilde{X}) = X$ of its G -orbit, and each morphism $f: \tilde{X} \rightarrow \tilde{Y}$ to the composite $X \xrightarrow{(g_1, \text{id}_{\tilde{X}})} \tilde{X} \xrightarrow{f} \tilde{Y} \xrightarrow{(g_2, \text{id}_Y)} Y$, where $\tilde{X} = g_1 \cdot X$ and $g_2 \cdot \tilde{Y} = Y$. We show that this equivalence is G -equivariant. Fix $h \in G$. Then $h \cdot \tilde{X}$ lies in the same G -orbit as $\tilde{X}$, and thus $E(h \cdot \tilde{X}) = E(\tilde{X})$.

Consider a morphism $(g, a): \tilde{X} \rightarrow \tilde{Y}$ in $A * G$, with $g_1 \cdot X = \tilde{X}$ and $g_2 \cdot \tilde{Y} = Y$. Then we have

$$E((g, a)) = (g_2, \text{id}_Y) \circ (g, a) \circ (g_1, \text{id}_{\tilde{X}}) = (g_2 g g_1, g_2 \cdot a).$$

We have $h \cdot (g, a) = (hgh^{-1}, h \cdot a): h \cdot \tilde{X} \rightarrow h \cdot \tilde{Y}$. Note that $(hg_1) \cdot X = h \cdot \tilde{X}$ and $(g_2 h^{-1}) \cdot (h \cdot \tilde{Y}) = Y$. Thus

$$E(h \cdot (g, a)) = (g_2 h^{-1}, \text{id}_Y) \circ (hgh^{-1}, h \cdot a) \circ (hg_1, \text{id}_{\tilde{X}}) = (g_2 g g_1, g_2 \cdot a) = E((g, a)),$$

showing the G -equivariance on morphisms. $\square$

We can always arrange the G -action on A to be free on the set of objects:

Lemma 3.4. *There exists a dg category A' with a G -action that is free on objects, together with a G -equivariant dg functor $A' \rightarrow A$ that is an equivalence of dg categories.*

Proof. The set $\text{ob}(A)$ of objects of A splits into the orbits of the G -action. We choose a set of representatives $R \subset \text{ob}(A)$ of the orbits $\text{ob}(A)/G$. We define the set of objects $\text{ob}(A')$ of A' to be pairs (x, g) with $x \in R$ and $g \in G$. The morphism complexes are defined as $\text{Map}_{A'}((x, g), (y, h)) = \text{Map}_A(g \cdot x, h \cdot y)$ with composition as in A . The apparent dg functor $\pi: A' \rightarrow A$, given on objects by $(x, g) \mapsto g \cdot x$ is fully faithful and essentially surjective, and hence an equivalence of dg categories. Using the G -action on A , we define a G -action on A' , given on objects by $h \cdot (x, g) = (x, hg)$ and on morphism complexes as for A . The equivalence of dg categories π is clearly G -equivariant. $\square$

Let A' be as in Lemma 3.4. Then $(A' * G)^{\text{red}}$ is by Lemma 3.3 the tip of a cocone under the diagram $\rho_{A'}: BG \rightarrow \text{dgCat}_k$ describing the G -action on A' . The (strict) G -equivariant equivalence of dg categories $A' \rightarrow A$ amounts to an equivalence of the corresponding (strict) diagrams $\rho_{A'}, \rho_A: BG \rightarrow \text{dgCat}_k$ and induces a dg equivalence $A' * G \rightarrow A * G$. Composing with the dg equivalence $(A' * G)^{\text{red}} \simeq A' * G$ and passing to derived ∞ -categories, we obtain equivalences in $\text{LinCat}_{\text{Mod}_k}$

$$\mathcal{D}((A' * G)^{\text{red}}) \simeq \mathcal{D}(A' * G) \simeq \mathcal{D}(A * G).$$

By composing with this equivalence, we can realize $\mathcal{D}(A * G)$ as the tip of a cocone under the diagram

$$\rho_{\mathcal{D}(A)} \simeq \rho_{\mathcal{D}(A')}: BG \rightarrow \text{LinCat}_{\text{Mod}_k}$$

describing the G -action on $\mathcal{D}(A)$. We denote the colimit of the functor $\rho_{\mathcal{D}(A)}$ by $\mathcal{D}(A)_G$. By its universal property, there is an induced functor

$$\zeta: \mathcal{D}(A)_G \longrightarrow \mathcal{D}(A * G). \quad (5)$$

Theorem 3.5. *The functor ζ from (5) is an equivalence of k -linear ∞ -categories.*

Remark 3.6. The ∞ -category underlying the model category dgCat_k with the Morita model structure is equivalent to the ∞ -category LinCat_k^ω of k -linear, compactly generated, presentable, and stable ∞ -categories as well as compact objects preserving, left adjoint functors, see [Coh13]. In particular, homotopy colimits in dgCat_k are described by ∞ -categorical colimits in LinCat_k^ω , see [Cis19, Rem. 7.9.10]. Note further that the forgetful functor $\text{LinCat}_k^\omega \rightarrow \text{LinCat}_k$ preserves and reflects colimits. Thus, by Theorem 3.5, $\text{Perf}(A * G)$ describes the homotopy colimit of ρ_A .

Proof of Theorem 3.5. By construction, there is a commutative diagram of compact objects preserving k -linear functors,

$$\begin{array}{ccc} & \mathcal{D}(A) & \\ F \swarrow & & \searrow \mathcal{D}(F_A) \\ \mathcal{D}(A)_G & \xrightarrow{\zeta} & \mathcal{D}(A * G) \end{array} \quad (6)$$

where F denotes the functor contained in the colimit diagram. Note that $\mathcal{D}(A)$ is generated under colimits by the objects arising from the objects in A , which are automatically compact. Similarly, the images of the objects in A under F and $\mathcal{D}(F_A)$ compactly generate, using that F^R is monadic by Lemma 2.11 and [Lur17, Prop. 4.7.3.14], respectively, that F_A is essentially surjective. It hence suffices to show that for all $X, Y \in A$, the morphism

$$\text{Mor}_{\mathcal{D}(A)_G}(F(X), F(Y)) \rightarrow \text{Mor}_{\mathcal{D}(A * G)}(\mathcal{D}(F_A)(X), \mathcal{D}(F_A)(Y)) \quad (7)$$

is an equivalence. We have equivalences

$$\text{Mor}_{\mathcal{D}(A)_G}(F(X), F(Y)) \simeq \text{Mor}_{\mathcal{D}(A)}(X, F^R F(Y)) \simeq \text{Mor}_{\mathcal{D}(A)}(X, \coprod_{g \in G} g.Y) \simeq \coprod_{g \in G} \text{Map}_A(g.X, Y), \quad (8)$$

where the last equivalence uses that $X \in \mathcal{D}(A)$ is compact, and

$$\text{Mor}_{\mathcal{D}(A * G)}(\mathcal{D}(F_A)(X), \mathcal{D}(F_A)(Y)) \simeq \text{Map}_{A * G}(\tilde{X}, \tilde{Y}) = \coprod_{g \in G} \text{Map}_A(g.X, Y). \quad (9)$$

By the commutativity of the diagram (6), the morphism (7) restricts to an equivalence on the $\text{Map}_A(X, Y)$ -summands of (8) and (9). The same is true for all these summands, which can be seen as follows:

The functors F is canonically G -equivariant, where we equip $\mathcal{D}(A)_G$ with the trivial G -action. For $h \in G$, this G -equivariance induces an equivalence $F(h.Y) \simeq F(Y)$. Using that

$\zeta \circ F \simeq \mathcal{D}(F_A)$, this in turn induces an equivalence $\mathcal{D}(F_A)(h.Y) \simeq \mathcal{D}(F_A)(Y)$. From this, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}(A)_G}(F(X), F(h.Y)) & \xrightarrow{\zeta} & \mathrm{Mor}_{\mathcal{D}(A * G)}(\mathcal{D}(F_A)(X), \mathcal{D}(F_A)(h.Y)) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{Mor}_{\mathcal{D}(A)_G}(F(X), F(Y)) & \xrightarrow{\zeta} & \mathrm{Mor}_{\mathcal{D}(A * G)}(\mathcal{D}(F_A)(X), \mathcal{D}(F_A)(Y)) \end{array} \quad (10)$$

The equivalence $\coprod_{g \in G} g.(h.Y) \simeq F^R F(h.Y) \simeq F^R F(Y) \simeq \coprod_{g \in G} g.Y$ maps the g -component of the coproduct to the (gh) -component via an equivalence, as follows from inspecting (4). The equivalence arising from the left vertical equivalence in (10)

$$\begin{aligned} \coprod_{g \in G} \mathrm{Map}_A(h^{-1}.(g.X), Y) &\simeq \coprod_{g \in G} \mathrm{Map}_A(g.X, h.Y) \\ &\simeq \mathrm{Mor}_{\mathcal{D}(A)_G}(F(X), F(h.Y)) \\ &\simeq \mathrm{Mor}_{\mathcal{D}(A)_G}(F(X), F(Y)) \\ &\simeq \coprod_{g \in G} \mathrm{Map}_A(g.X, Y) \end{aligned}$$

thus also respects the components of the coproduct, in the sense that it maps the g -component to the $h^{-1}g$ -component via an equivalence.

The equivalence $\coprod_{g \in G} \mathrm{Map}_A(h^{-1}.(g.X), Y) \simeq \coprod_{g \in G} \mathrm{Map}_A(g.X, Y)$ arising from the right vertical equivalence in (10) similarly respects the components of the G -coproducts, as follows from inspecting the proof of Lemma 3.3 (which is the central step in the construction of the cocone).

Applying the previous argument to $h.X$ and Y instead of X, Y , we thus find that (7) restricts also to an equivalence on the $\mathrm{Map}_A(h.X, Y)$ summand. Since this holds for all $h \in G$, we conclude that (7) is an equivalence. $\square$

Remark 3.7. In the case that A is an A_∞ -category equipped with a strict group action as in [OZ22, AP26], the above construction easily adapts to show that the skew group A_∞ -category $A * G$ of [OZ22, AP26] is equivalent to the ∞ -categorical colimit over BG in the ∞ -category $\mathrm{Cat}_{A_\infty}[M^{-1}]$ of A_∞ -categories localized at Morita equivalences. For this, we use the equivalences of ∞ -categories $\mathrm{Cat}_{A_\infty}[M^{-1}] \simeq \mathrm{dgCat}_k[M^{-1}] \simeq \mathrm{LinCat}_k^\omega$, see [Pas24, COS24] and [Coh13].

Proof of Corollary 1.3. Suppose that A is a k -algebra, considered as a dg category with a single object. Using Theorem 3.5 and Lemma 2.7 we obtain an equivalence of ∞ -categories

$$\mathcal{D}(AG) \simeq \lim_{BG} \mathcal{D}(A) = \mathcal{D}(A)^G.$$

The functor $\mathcal{D}(A) \rightarrow \mathcal{D}(A)_G \simeq \mathcal{D}(A)^G$ from the colimit cone identifies under the equivalence $\mathcal{D}(AG) \simeq \mathcal{D}(A)_G$ with the pushforward along the algebra morphism $A \hookrightarrow AG$ considered in Lemma 3.1. Passing to right adjoints, we find that the morphism $\mathcal{D}(AG) \simeq \mathcal{D}(A)^G \rightarrow \mathcal{D}(A)$ from the limit cone identifies with the pullback along $A \hookrightarrow AG$.

By Remark 3.8 below, $N(\mathrm{Mod}(A)^G)$ embeds fully faithfully into $\mathcal{D}(A)^G$. The nerve $N(\mathrm{Mod}(AG))$ also embeds fully faithfully into $\mathcal{D}(AG)$ as the standard heart. It remains to note that these two full subcategories are identified under the above equivalence.

An object in $\mathcal{D}(AG)$ lies in the heart if and only if its image under the functor $\mathcal{D}(AG) \rightarrow \mathcal{D}(A)$ lies in the heart of $\mathcal{D}(A)$. An object in $\mathcal{D}(A)^G$ corresponds to a coCartesian section of the Grothendieck construction over BG [Lur24, Prop. 05RX], and lies in the image of $N(\mathrm{Mod}(A)^G)$ if and only if its evaluation at the unique object $* \in BG$ lies in $N(\mathrm{Mod}(A)) \subset \mathcal{D}(A)$. The evaluation functor at $* \in BG$ on coCartesian sections describes the functor $\mathcal{D}(A)^G \rightarrow \mathcal{D}(A)$ from the limit cone. Hence, both full subcategories have the same essential images.

The equivalence clearly restricts to the subcategories of finite modules (since finiteness is detected under the functors to $\mathcal{D}(A)$). $\square$

Remark 3.8. We describe the relation of the 1-category of equivariant objects with limits in the ∞ -category Cat_∞ of ∞ -categories. Let Cat denote the 1-category of 1-categories. Consider the adjunction

$$h(-): \text{Set}_\Delta \leftrightarrow \text{Cat} : N(-)$$

between the homotopy category functor and the nerve functor. Note that $N(-)$ is fully faithful. This adjunction is a Quillen adjunction with respect to the Joyal model structure on Set_Δ and the standard model structure on Cat . We hence obtain an adjunction between ∞ -categories $h(-): \text{Cat}_\infty \leftrightarrow \text{Cat}[W^{-1}] : N(-)$, where W denotes the collection of equivalences of 1-categories. The derived functor $N(-)$ is also fully faithful. Since it preserves ∞ -categorical limits, we see that limits in Cat_∞ of diagrams valued in nerves of 1-categories are equivalent to nerves of 1-categories, and hence determined by their homotopy categories. Given a diagram $\rho: BG \rightarrow \text{Cat}, * \rightarrow C$, one finds that the homotopy category of the limit of $N \circ \rho: BG \rightarrow \text{Cat}_\infty$ is equivalent to the category C^G of G -equivariant objects of C , which can be seen using the explicit description of limits via coCartesian sections of the Grothendieck construction [Lur24, Prop. 05RX]. Thus, we have

$$N(C^G) \simeq \lim_{BG} N \circ \rho \in \text{Cat}[W^{-1}] \subset \text{Cat}_\infty.$$

4 Orbit categories

In this section, we will specialize to the case where $G = \mathbb{Z}$. The arising group quotients are known as orbit categories. After a general discussion in the ∞ -categorical context in Section 4.1, we consider non-strict $\mathbb{Z}$ -actions on dg categories in Section 4.2. Finally, we describe examples arising from periodic derived categories in Section 4.3.

4.1 Orbit ∞ -categories

Let L denote the simplicial set with a unique 0-simplex $*$ and a unique non-degenerate 1-simplex $1: * \rightarrow *$.

Definition 4.1. Let $F: \mathcal{C} \rightarrow \mathcal{C}$ be a Mod_R -linear endofunctor of a Mod_R -linear ∞ -category $\mathcal{C}$. The Mod_R -linear orbit ∞ -category $\mathcal{C}/F$ is defined as the colimit of the functor

$$L \longrightarrow \text{LinCat}_{\text{Mod}_R}, \quad * \mapsto \mathcal{C}, \quad (1: * \rightarrow *) \mapsto F. \quad (11)$$

Remark 4.2. The orbit ∞ -category $\mathcal{C}/F$ is also equivalent to the limit of the functor obtained from (11) by passing to right adjoints. The limit can be described as the ∞ -category of coCartesian sections of the Grothendieck construction. An object of $\mathcal{C}/F$ thus amounts to an object $X \in \mathcal{C}$ together with an equivalence $X \simeq F^R(X)$ in $\mathcal{C}$.

Lemma 4.3. Let $F: \mathcal{C} \rightarrow \mathcal{C}$ be a Mod_R -linear endofunctor.

- (1) The inclusion of simplicial sets $L \subset BN, 1 \mapsto 1$ is inner anodyne. Pulling back along this inclusion hence gives rise to a commutative diagram of ∞ -categories:

$$\begin{array}{ccc} \text{Fun}(BN, \text{LinCat}_{\text{Mod}_R}) & \xrightarrow{\simeq} & \text{Fun}(L, \text{LinCat}_{\text{Mod}_R}) \\ & \searrow \text{colim} & \swarrow \text{colim} \\ & \text{LinCat}_{\text{Mod}_R} & \end{array}$$

- (2) Pulling back along the inclusion $BN \subset B\mathbb{Z}$ yields a commutative diagram of ∞ -categories,

$$\begin{array}{ccc} \text{Fun}(B\mathbb{Z}, \text{LinCat}_{\text{Mod}_R}) & \xleftarrow{\quad} & \text{Fun}(BN, \text{LinCat}_{\text{Mod}_R}) \\ & \searrow \text{colim} & \swarrow \text{colim} \\ & \text{LinCat}_{\text{Mod}_R} & \end{array}$$

where the horizontal functor is fully faithful. Its essential image consists of those functors $BN \rightarrow \text{LinCat}_{\text{Mod}_R}$ mapping 1 to an equivalence.

In particular, the orbit ∞ -category $\mathcal{C}/F$ of a Mod_R -linear equivalence $F: \mathcal{C} \rightarrow \mathcal{C}$ is equivalent to the colimit over a functor $B\mathbb{Z} \rightarrow \text{LinCat}_{\text{Mod}_R}$, mapping 1 to F .

Proof. We begin with showing part (1). We can consider L as a direct graph in the sense of [Lur24, Def. 001E]. Then $B\mathbb{N}$ is isomorphic to the nerve of its path category in the sense of [Lur24, Constr. 00HK]. That the inclusion $L \subset B\mathbb{N}$ is inner anodyne is thus proved in [Lur24, Prop. 00J6]. The inclusion $L \subset B\mathbb{N}$ is therefore final and cofinal by [Lur09, Prop. 4.1.1.3], showing the commutativity of the diagram in (1). The horizontal functor is an equivalence by [Lur24, Cor. 01EJ] and [Lur24, Prop. 01EF].

We next show part (2). The inclusion $B\mathbb{N} \subset B\mathbb{Z}$ is a weak homotopy equivalence and, using that $B\mathbb{Z}$ is a Kan complex, hence final and cofinal [Lur09, Cor. 4.1.2.6]. This shows the commutativity of the diagram in (2). The fully faithfulness of the horizontal functor amounts to the statement that $B\mathbb{Z}$ is the localization of $B\mathbb{N}$ at the morphism 1. Since the morphism 1 generates all non-identity morphisms in $B\mathbb{N}$ under composition, this is equivalent to the assertion that $B\mathbb{Z}$ is the localization of $B\mathbb{N}$ at all morphisms. This directly follows from [Lur24, Prop. 01MY]. $\square$

4.2 Orbit dg categories

We assume that k is a field. Given a dg category A , we denote by $\text{rep}_{\text{dg}}(A, A)$ the dg category of cofibrant dg A - A -bimodules which are right quasi-representable, see also [Kel06, FKQ24] for detailed definitions. We note that any such bimodule $F \in \text{rep}_{\text{dg}}(A, A)$ gives rise to a dg functor $\text{Perf}(F): \text{Perf}(A) \rightarrow \text{Perf}(A)$, and thus to a Mod_k -linear functor $\mathcal{D}(F): \mathcal{D}(A) \rightarrow \mathcal{D}(A)$ between the derived ∞ -categories. We call $F \in \text{rep}_{\text{dg}}(A, A)$ a Morita equivalence if the dg functor $\text{Perf}(F)$ is a quasi-equivalence (or equivalently $\mathcal{D}(F)$ is an equivalence of ∞ -categories).

Given $F \in \text{rep}_{\text{dg}}(A, A)$, [FKQ24] defines the dg orbit category $A/F^{\mathbb{Z}}$ as a dg localization of the 'left lax quotient dg category' $A/_U F^{\mathbb{N}}$, which is the dg category with the same objects as A and morphism complexes $\text{Map}_{A/_U F^{\mathbb{N}}}(X, Y) = \coprod_{i \in \mathbb{N}} \text{Map}_{\text{Perf}(A)}(X, F^i(Y))$.

Remark 4.4. Supposing that $F \in \text{rep}_{\text{dg}}(A, A)$ is a Morita equivalence, one can show that, for all $X, Y \in A$, there is a quasi-isomorphism

$$\text{Map}_{A/F^{\mathbb{Z}}}(X, Y) \simeq \coprod_{i \in \mathbb{Z}} \text{Map}_{\text{Perf}(A)}(X, F^i(Y)). \quad (12)$$

Proposition 4.5. *Let $F \in \text{rep}_{\text{dg}}(A, A)$ be a Morita equivalence. Then there exists a canonical equivalence of Mod_k -linear ∞ -categories*

$$\mathcal{D}(A)/\mathcal{D}(F) \simeq \mathcal{D}(A/F^{\mathbb{Z}})$$

between the Mod_k -linear orbit ∞ -category and the derived ∞ -category of the dg orbit category.

Proof of Proposition 4.5. As shown in [FKQ24, Section 3.2], there exists a dg functor $\pi: A \rightarrow A/F^{\mathbb{Z}}$ together with a natural equivalence $\pi \circ F \simeq \pi$. Passing to compact, cofibrant modules, F induces a dg functor $\text{Perf}(F): \text{Perf}(A) \rightarrow \text{Perf}(A)$ (and not just a dg bimodule) together with a natural equivalence $\text{Perf}(\pi) \circ \text{Perf}(F) \simeq \text{Perf}(\pi)$. This exhibits $\text{Perf}(A/F^{\mathbb{Z}})$ as the tip of a cocone under the functor $L \rightarrow \text{dgCat}_k[M^{-1}]$, $1 \mapsto \text{Perf}(F)$. Passing to derived ∞ -categories, this induces a Mod_k -linear functor $\zeta: \mathcal{D}(A)/\mathcal{D}(F) \rightarrow \mathcal{D}(A/F^{\mathbb{Z}})$. This functor is checked to be an equivalence using an analogous argument as in the proof of Theorem 3.5: Denote by $p: \mathcal{D}(A) \rightarrow \mathcal{D}(A)/\mathcal{D}(F)$ the functor from the colimit cone. For $X, Y \in A$, it suffices to show that the morphism

$$\begin{aligned} \coprod_{i \in \mathbb{Z}} \text{Map}_{\text{Perf}(A)}(X, F^i(Y)) &\simeq \text{Mor}_{\mathcal{D}(A)/\mathcal{D}(F)}(p(X), p(Y)) \\ &\xrightarrow{\zeta} \text{Mor}_{\mathcal{D}(A/F^{\mathbb{Z}})}(\mathcal{D}(\pi)(X), \mathcal{D}(\pi)(Y)) \\ &\simeq \coprod_{i \in \mathbb{Z}} \text{Map}_{\text{Perf}(A)}(X, F^i(Y)) \end{aligned}$$

is an equivalence, for which we show that it restricts to an equivalence on each component of the coproduct. For $j = 0$, this follows as in Theorem 3.5. The functors $p: \mathcal{D}(A) \rightarrow \mathcal{D}(A)/\mathcal{D}(F)$ and $\mathcal{D}(\pi): \mathcal{D}(A) \rightarrow \mathcal{D}(A/F^\mathbb{Z})$ are again $\mathbb{Z}$ -equivariant with respect to the trivial $\mathbb{Z}$ -action on the targets. For $j > 0$, the argument that the equivalence restricts on the j -th component is analogous to the argument in the proof of Theorem 3.5 around diagram (10), using the description of the $\mathbb{Z}$ -equivariance datum of π given in [FKQ24], based on the construction of the natural transformation in [FKQ24, Eq. (3.6)]. In the case $j < 0$, we can again apply the above argument for $-j > 0$, starting with $F^j(X)$ instead of X . $\square$

Remark 4.6. Suppose that $F: A \rightarrow A$ is a dg functor. An a priori different definition of orbit dg category is given in [Kel05], which we denote by $A/\mathrm{dg} F$. There is an apparent diagram of dg categories

$$\begin{array}{ccc} A & \xrightarrow{F} & A \\ & \searrow & \swarrow \\ & A/\mathrm{dg} F & \end{array}$$

where the natural transformation evaluates at $X \in A$ to the canonical morphism $F(X) \rightarrow X$ in $A/\mathrm{dg} F$. Using the universal property shown in [FKQ24], this induces a dg functor $A/\mathrm{d}F^\mathbb{N} \rightarrow A/\mathrm{dg} F$, which in turn induces a dg functor $\alpha: A/F^\mathbb{Z} \rightarrow A/\mathrm{dg} F$. If F is a Morita equivalence, the quasi-isomorphisms (12) imply that α is a quasi-equivalence.

In the case that F is strictly invertible, the orbit dg category $A/\mathrm{dg} F$ is furthermore isomorphic to the skew group dg category $A * \mathbb{Z}$ from Section 3.

4.3 Example: Periodic derived categories

We fix a field k . Let $\mathrm{Perf}(k)$ be the dg category of chain complexes over k with finite dimensional total homology. We further fix $n \geq 1$ and consider the invertible endofunctor $[n]: \mathrm{Perf}(k) \rightarrow \mathrm{Perf}(k)$ given by the shift functor. This defines a $\mathbb{Z}$ -action on $\mathrm{Perf}(k)$, with orbit dg category $\mathrm{Perf}(k)/[n]$.

Let $k[t_n]$ be the dg algebra of graded polynomials with the monomial t_n in degree n (in homological grading). Note that $k[t_n]$ is the $(n+1)$ -Calabi–Yau completion of k in the sense of [Kel11]. We similarly denote by $k[t_n^\pm]$ the dg algebra of graded Laurent polynomials.

Proposition 4.7. *There are equivalences in $\mathrm{LinCat}_{\mathrm{Mod}_k}$*

$$\mathcal{D}(\mathrm{Perf}(k)/[n]) \simeq \mathcal{D}(k)/[n] \simeq \mathcal{D}(k[t_n^\pm]).$$

Proof. The first equivalence is the statement of Proposition 4.5. For the second equivalence, we produce a functor $\mathcal{D}(k)/[n] \rightarrow \mathcal{D}(k[t_n^\pm])$ using the universal property of the colimit. We employ the following trick, to avoid any discussions about signs for the shift functor: any Mod_k -linear functor $\mathcal{D}(k) \rightarrow \mathcal{D}(k[t_n^\pm])$ is fully determined by the image of k . Since $k[t_n^\pm][n] \simeq k[t_n^\pm] \in \mathcal{D}(k[t_n^\pm])$, we thus find a commutative diagram in $\mathrm{LinCat}_{\mathrm{Mod}_k}$:

$$\begin{array}{ccc} \mathcal{D}(k) & \xrightarrow{[n]} & \mathcal{D}(k) \\ & \searrow & \swarrow \\ & \mathcal{D}(k[t_n^\pm]) & \end{array}$$

$k \mapsto k[t_n^\pm] \quad k \mapsto k[t_n^\pm]$

This induces a functor $\mathcal{D}(k)/[n] \rightarrow \mathcal{D}(k[t_n^\pm])$, which is now readily checked to be fully faithful and essentially surjective. $\square$

Remark 4.8. Let $\mathcal{D}^{\mathrm{fin}}(k[t_n]) \subset \mathcal{D}^{\mathrm{perf}}(k[t_n])$ denote the full subcategory of modules with finite dimensional total homology. There is an equivalence in $\mathrm{LinCat}_{\mathrm{Mod}_k}$

$$\mathcal{D}(k[t_n])/ \mathrm{Ind} \mathcal{D}^{\mathrm{fin}}(k[t_n]) \simeq \mathcal{D}(k[t_n^\pm]),$$

see [Chr22, Lem. 2.11]. By [HI22, Rem. 2.14.(2)], there is an equivalence of dg categories

$$\mathrm{Perf}(k)/[n] \simeq \mathrm{Perf}(k[t_n]) / \mathrm{Perf}(k[t_n])^{\mathrm{fin}},$$

where $\mathrm{Perf}(k[t_n])^{\mathrm{fin}}$ denotes the full dg subcategory of $\mathrm{Perf}(k[t_n])$ of dg modules with finite dimensional total homology. Combining these equivalences provides an alternative way to prove Proposition 4.7.

Lemma 4.9. *Let $\mathcal{C}$ be a Mod_k -linear ∞ -category. Then the orbit category $\mathcal{C}/[n]$ is equivalent to the tensor product $\mathcal{C} \otimes_{\mathrm{Mod}_k} \mathcal{D}(k[t_n^\pm])$ in $\mathrm{LinCat}_{\mathrm{Mod}_k}$.*

Proof. Using that $\mathcal{D}(k) \simeq \mathrm{Mod}_k$ and that the tensor product in $\mathrm{LinCat}_{\mathrm{Mod}_k}$ preserves colimits in the second entry, we find equivalences

$$\mathcal{C} \otimes_{\mathrm{Mod}_k} \mathcal{D}(k[t_n^\pm]) \simeq \mathcal{C} \otimes_{\mathrm{Mod}_k} \mathrm{colim}_L \mathrm{Mod}_k \simeq \mathrm{colim}_L \mathcal{C} = \mathcal{C}/[n]. \quad \square$$

The above allows us to obtain the following concrete description of the 1-periodic topological Fukaya category, or equivalently Higgs category, of a marked surface studied in [Chr22]. Note that the 2-Calabi–Yau cluster category of a marked surface is a localization of this Higgs category [Wu23], and the following result thus establishes a precise relationship between derived categories of gentle algebras and the cluster categories of surfaces.

Proposition 4.10. *Let A be a finite dimensional gentle algebra and $\mathbf{S}$ the corresponding marked surface in the sense of [OPS18]. Assume that every boundary component of $\mathbf{S}$ contains at least one marked point⁴. Then the Higgs category $\mathcal{H}$ [Wu23] associated with $\mathbf{S}$, described in [Chr22], is equivalent to the perfect derived ∞ -category of the orbit dg category $\mathrm{Perf}(A)/[1]$.*

In particular, the 1-categorical orbit category $(\mathrm{ho} \mathrm{Perf}(A))/[1]$ embeds fully faithfully into the triangulated homotopy 1-category $\mathrm{ho} \mathcal{H}$ of the Higgs category.

Proof of Proposition 4.10. It is shown in [Chr22] that the Ind-completion of the ∞ -categorical Higgs category $\mathcal{H}$ is equivalent to $\mathcal{D}(A) \otimes_{\mathrm{Mod}_k} \mathcal{D}(k[t_1^\pm])$ for a gentle algebra A , see Proposition 4.16 and Theorem 8.4 in loc. cit.. Restricting to compact objects yields the desired equivalence. The identification of the gentle algebra A , as being associated with the same marked surface $\mathbf{S}$ as $\mathcal{H}$ can be seen as follows. There is an equivalence between $\mathcal{D}(A)$ and the (Ind-complete) partially wrapped Fukaya category of $\mathbf{S}$ (equipped with an induced choice of framing) stated on page 4 of [OPS18], that is based on [HKK17]. This partially wrapped Fukaya category arises as the global sections of a non-singular perverse schober with generic stalk $\mathcal{D}(k)$ and trivial monodromy, as noted on page 5 of [Chr26], based on the cosheaf description of the partially wrapped Fukaya category given in [HKK17]. $\square$

5 Spectral skew group algebras

In this final section, we introduce a spectral version of the skew group algebra and generalize Theorem 3.5 to the spectral setting.

We fix a base $\mathbb{E}_\infty$ -ring spectrum R . Consider a Mod_R -linear ∞ -category $\mathcal{C}$ with a G -action. In the case that $\mathcal{C} \simeq R\mathrm{Mod}_A$ has a compact generator A , the functor $\mathrm{Mor}_{\mathcal{C}}(A, -): \mathcal{C} \rightarrow \mathrm{Mod}_R$ is conservative, as is the functor $\mathcal{C}_G \rightarrow \mathcal{C}$ by Lemma 2.11. Thus the composite $\mathcal{C}_G \rightarrow \mathcal{C} \rightarrow \mathrm{Mod}_R$ is also conservative. The functor $\mathcal{C} \rightarrow \mathcal{C}_G$ preserves compact objects since it admits a colimit preserving right adjoint $\psi^R \otimes_{\mathrm{Mod}_R^{H_G}} \mathcal{C}$ by Example 2.5 and Lemma 2.10. The image of R under the left adjoint $\mathrm{Mod}_R \rightarrow \mathcal{C}_G$ defines a compact generator Y of $\mathcal{C}_G$. In the case that A is fixed by the G -action, the R -linear endomorphism algebra of Y can be considered as the corresponding skew group algebra AG . To be able to describe the spectral skew group algebra AG , we give a more systematic construction of it in Definition 5.3.

⁴This is equivalent to the assertion that A is smooth, which follows from [LP20, Lem. 3.1.3].

Construction 5.1. Let $\text{Mod}_R \in \text{Fun}(BG, \text{LinCat}_{\text{Mod}_R})$ denote the constant functor with value Mod_R . We define the functor ξ as the following composite:

$$\begin{aligned} \text{Fun}(BG, (\text{LinCat}_{\text{Mod}_R})_{\text{Mod}_R /}) &\rightarrow \text{Fun}(BG, \text{LinCat}_{\text{Mod}_R})_{\text{Mod}_R /} \\ &\rightarrow \text{Fun}(BG, \text{LinCat}_{\text{Mod}_R})_{\text{Mod}_R^{\text{IG}} /} \\ &\simeq \left(\text{LinCat}_{\text{Mod}_R^{\text{IG}}} \right)_{\text{Mod}_R^{\text{IG}} /} . \end{aligned}$$

The G -action on Mod_R^{IG} is given by permuting the G -components, so that $\text{colim}_{BG} \text{Mod}_R^{\text{IG}} \simeq \text{Mod}_R \in \text{LinCat}_{\text{Mod}_R}$. This equivalence is adjoint to a morphism $\text{Mod}_R^{\text{IG}} \rightarrow \text{Mod}_R$ in $\text{Fun}(BG, \text{LinCat}_{\text{Mod}_R})$ which is used for the second functor above. The third functor uses the equivalence from Proposition 2.8.

Proposition 5.2. *There is a commutative diagram of ∞ -categories*

$$\begin{array}{ccccc} \text{Fun}(BG, \text{Alg}(\text{Mod}_R)) & \xrightarrow{\text{Fun}(BG, \Theta_*)} & \text{Fun}(BG, (\text{LinCat}_{\text{Mod}_R})_{\text{Mod}_R /}) & \longrightarrow & \text{Fun}(BG, \text{LinCat}_{\text{Mod}_R}) \\ \downarrow & & \downarrow \xi & & \downarrow \simeq \\ \text{Alg}(\text{Mod}_R^{\text{IG}}) & \xrightarrow{\Theta_*} & \left(\text{LinCat}_{\text{Mod}_R^{\text{IG}}} \right)_{\text{Mod}_R^{\text{IG}} /} & \longrightarrow & \text{LinCat}_{\text{Mod}_R^{\text{IG}}} \end{array}$$

where the right vertical functor is the equivalence from Proposition 2.8. We denote the left vertical functor by

$$\xi^{\text{Alg}}: \text{Fun}(BG, \text{Alg}(\text{Mod}_R)) \rightarrow \text{Alg}(\text{Mod}_R^{\text{IG}}).$$

Proof. The commutativity of the right diagram follows from the definition of ξ . Using the fully faithfulness of the left horizontal morphisms, we can induce the left vertical functor from ξ by proving that objects in the image of $\text{Fun}(BG, \Theta_*)$ are mapped by ξ to elements in the image of Θ_* .

Let $\rho_A: BG \rightarrow \text{Alg}(\text{Mod}_R)$, $* \mapsto A$. Then $\xi \circ \text{Fun}(BG, \Theta_*)(\rho_A)$ is given by a Mod_R^{IG} -linear functor $F: \text{Mod}_R^{\text{IG}} \rightarrow \text{RMod}_A$, component-wise given by $(-) \otimes A$, i.e. satisfying $F(R_g) \simeq A$ for all $g \in G$ ⁵.

We apply [Lur17, Prop. 4.8.5.8] to show that the functor $F: \text{Mod}_R^{\text{IG}} \rightarrow \text{RMod}_A$ lies in the image of Θ_* , thus showing the existence of the left commutative square. Conditions (1), (2), (3) and (5) are clear. For condition (4), it suffices to observe that the right adjoint F^R of $F: \text{Mod}_R^{\text{IG}} \rightarrow \text{RMod}_A$ is G -componentwise a functor that preserves geometric realizations, and thus preserves geometric realizations as well. Condition (6) boils down to the statement that F^R and the counit of $F \dashv F^R$ commute with the Mod_R^{IG} -action, which is straightforward to check. Note that for F^R , this follows from Lemma 2.4.(2). $\square$

Definition 5.3. Consider an action $\rho: BG \rightarrow \text{Alg}(\text{Mod}_R)$ of the group G on an R -linear ring spectrum A . The skew group algebra $AG \in \text{Alg}(\text{Mod}_R)$ is defined as the image of ρ under the functors

$$\text{Fun}(BG, \text{Alg}(\text{Mod}_R)) \xrightarrow{\xi^{\text{Alg}}} \text{Alg}(\text{Mod}_R^{\text{IG}}) \xrightarrow{\text{Alg}(\psi)} \text{Alg}(\text{Mod}_R) \quad (13)$$

from Remark 2.3 and Proposition 5.2.

Remark 5.4. We unravel the construction of the skew group algebra AG .

Starting with an R -linear ring spectrum A , with the group G acting on it, we have an induced G -action on its ∞ -category of (right) modules RMod_A , which we can also consider as a left-tensoring of RMod_A by Mod_R^{IG} . Considering A as an object of RMod_A , we consider the

⁵We point out the curious fact that the functors F and F^R only depend on the Mod_R -linear structure of $\mathcal{C}$, and not on its Mod_R^{IG} -linear structure, meaning the G -action on $\mathcal{C}$. This is because A is fixed by the G -action on $\mathcal{C}$. The G -action will only contribute when describing the Mod_R^{IG} -linear endomorphism algebra structure of $F^R(A)$, see Remark 5.4 below.

$\text{Mod}_R^{\text{II}G}$ -linear endomorphism algebra $\tilde{A} \in \text{Alg}(\text{Mod}_R^{\text{II}G})$ of A , in the sense of [Lur17, Section 4.7.1]. The algebra $\tilde{A}$ is a refinement of the skew group algebra AG , which is defined as the image of $\tilde{A}$ under the monoidal functor $\psi = \coprod: \text{Mod}_R^{\text{II}G} \rightarrow \text{Mod}_R$. It remains to unravel its definition to describe the skew group algebra more explicitly.

Given $g \in G$, the restriction to the corresponding component $\text{Mod}_R \subset \text{Mod}_R^{\text{II}G}$ of $\tilde{A}$ is given by A . Inspecting the definition, one sees that this amounts to the fact that A is preserved by the action of g . We write this decomposition as $\tilde{A} = \coprod_{g \in G} \tilde{A}_g$. Note that we thus have $AG \simeq \coprod_{g \in G} A$.

We turn to the algebra structure of $\tilde{A}$ (describing it on the level of homotopy groups). To make the notation for this more transparent, we denote the $\text{Mod}_R^{\text{II}G}$ -action by $\otimes^G$. Firstly, we consider $\tilde{A}$ as equipped with the map $m: \tilde{A} \otimes^G A \rightarrow A$ in RMod_A , that is simply the multiplication map of A on every component $A \otimes A \simeq \tilde{A}_g \otimes^G A$. The crucial point of the construction of $\tilde{A}$ in [Lur17, Section 4.7.1] is that the multiplication map $\tilde{m}: \tilde{A} \otimes^G \tilde{A} \rightarrow \tilde{A}$ in $\text{Mod}_R^{\text{II}G}$ is uniquely determined by the property that the following diagram commutes⁶:

$$\begin{array}{ccc} \tilde{A} \otimes^G \tilde{A} \otimes^G A & \xrightarrow{\tilde{A} \otimes^G m} & \tilde{A} \otimes^G A \\ \downarrow \tilde{m} \otimes^G A & & \downarrow m \\ \tilde{A} \otimes^G A & \xrightarrow{m} & A \end{array}$$

We write elements of the homotopy groups of $\tilde{A}$ as pairs (g, a) with $g \in G$ and $a \in \pi_n(A)$. We next argue that the morphism

$$\tilde{A} \otimes^G m: \tilde{A}_g \otimes^G \tilde{A} \otimes^G A \rightarrow \tilde{A}_g \otimes^G A$$

is given on pure tensors and on the level of homotopy groups by

$$(g, a) \otimes (h, b) \otimes c \mapsto (g, a) \otimes ((g.b) \cdot c).$$

The above identity follows from the following three observations:

- Every element $(h, b) \in \pi_n(\tilde{A})$ determines a morphism $A \rightarrow \tilde{A} \otimes^G A[-n]$ in RMod_A (up to equivalence), given on the level of homotopy groups by mapping c to the pure tensor $(h, b) \otimes c$.
- For every $(h, b) \in \pi_n(\tilde{A})$, the following diagram commutes:

$$\begin{array}{ccc} A & & \\ c \mapsto (h, b) \otimes c \downarrow & \searrow b \cdot (-) & \\ \tilde{A} \otimes^G A[-n] & \xrightarrow{m} & A[-n] \end{array}$$

- For every $b \in \pi_n(A)$ the tensor product $R_g \otimes^G (-)$ with the object $R_g = R$ lying in the g -component of $\text{Mod}_R^{\text{II}G}$ maps the morphism $A \xrightarrow{b \cdot (-)} A[-n]$ to the morphism $A \xrightarrow{g.b \cdot (-)} A[-n]$, where g refers to the G -action on $\pi_n(A)$.

With the above, it is now straightforward to see that $\tilde{m}$ is given on pure tensors and homotopy groups by

$$\tilde{m}: \tilde{A} \otimes^G \tilde{A} \rightarrow \tilde{A}, \quad (g, a) \otimes (h, b) \mapsto (gh, a \cdot (g.b)),$$

matching the desired formula of the multiplication of the skew group algebra.

In case that A is a discrete algebra object of Mod_k for a commutative ring k , we thus see that the skew group algebra AG is quasi-isomorphic to the classical skew group algebra.

Finally, we show that the module ∞ -category of AG describes the group quotient.

⁶This follows from the fact that $m: \tilde{A} \otimes^G A \rightarrow A$ is a terminal object in the monoidal ∞ -category $\text{Mod}_R^{\text{II}G}[A]$ described in [Lur17, Section 4.7.1].

Proof of Theorem 1.2. Let $\tilde{A} \in \text{Alg}(\text{Mod}_R^{\text{HG}})$ be the image of ρ under the functor ξ^{Alg} . We have equivalences

$$\begin{aligned} \text{colim}_{BG} \text{RMod}_A &\simeq \psi_!(\text{RMod}_A) \\ &\simeq \psi_!(\text{RMod}_{\tilde{A}}(\text{Mod}_R^{\text{HG}})) \\ &\simeq \text{RMod}_{\psi(\tilde{A})} = \text{RMod}_{AG}, \end{aligned}$$

where the first equivalence follows from Lemma 2.10, the second from the commutativity of the left square in Proposition 5.2, and the third equivalence follows from the statement in [Lur17, Prop. 4.8.5.1] that the functor Θ preserves coCartesian edges, see also the last paragraph in the proof of that statement. $\square$

Remark 5.5. If A is a dg algebra with a strict G -action, we can deduce from combining Theorem 1.2 and Theorem 3.5 an equivalence of k -linear ∞ -categories $\text{RMod}_{AG} \simeq \mathcal{D}(A * G)$ under $\mathcal{D}(A) \simeq \text{RMod}_A$. The functors $\mathcal{D}(A) \rightarrow \text{RMod}_{AG}, \mathcal{D}(A * G)$ map A to AG , respectively, $A * G$. Passing to the (k -linear ∞ -categorical) endomorphism algebras, we find that the dg categorical skew group algebra $A * G$ is quasi-isomorphic to AG .

Remark 5.6. We expect that the results of this paper can be generalized to G being a group object in spaces.

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